Representing Edge Flows via Sparse Cell Complexes

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Learning on Graphs – November 29, 2023

RWTH Aachen University

Funded by the European Union
European Research Council
Established by the European Commission
Computational Network Science

Motivation

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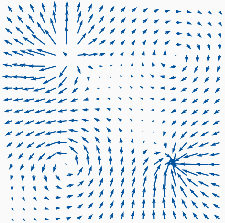
Intuition – Interpreting complicated Data



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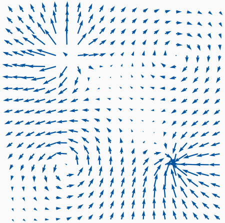

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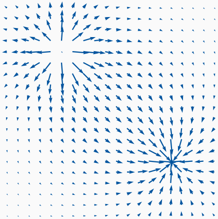

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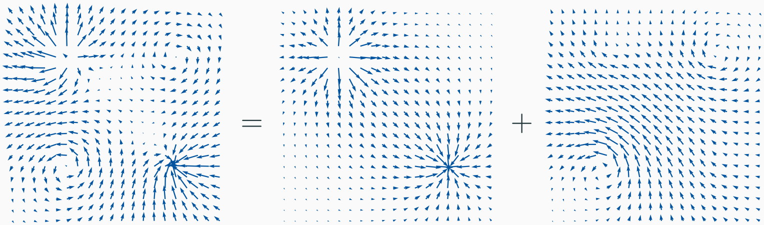

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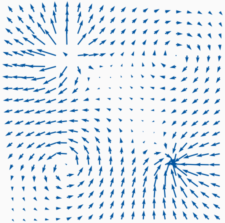

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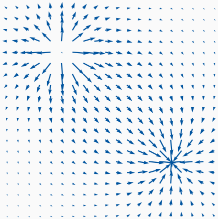

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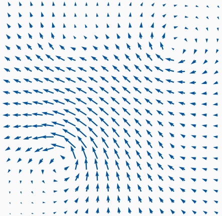

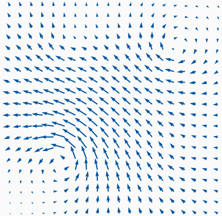

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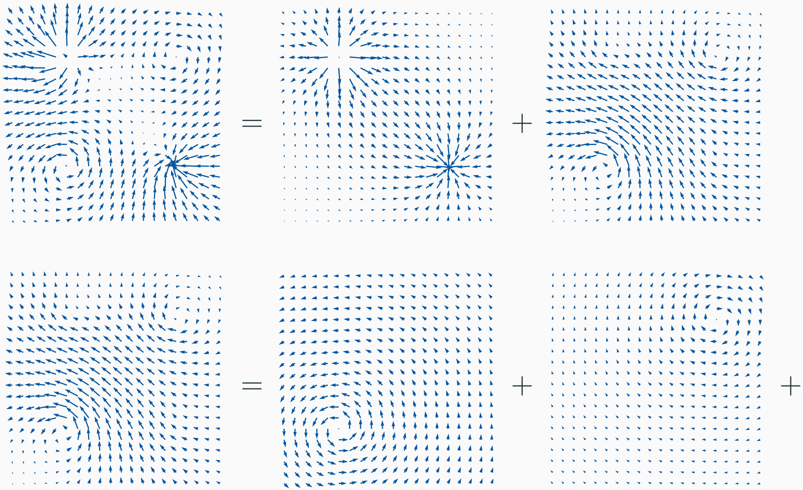

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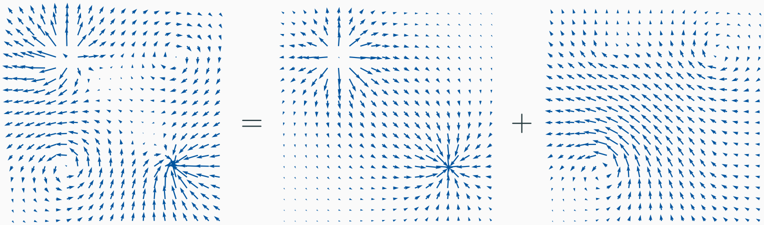
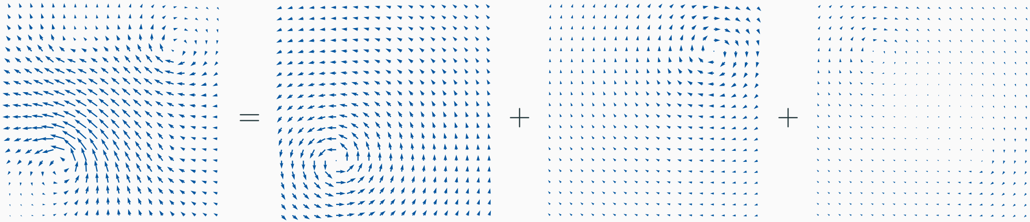
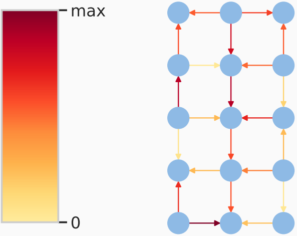




Topological Signal Processing: The Hodge Decomposition



edge flows

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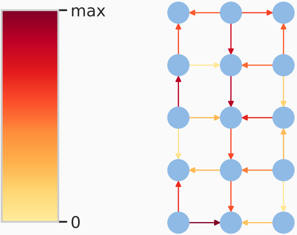

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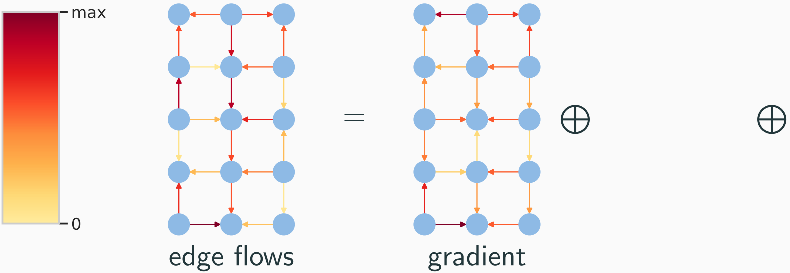

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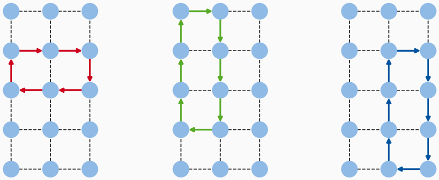

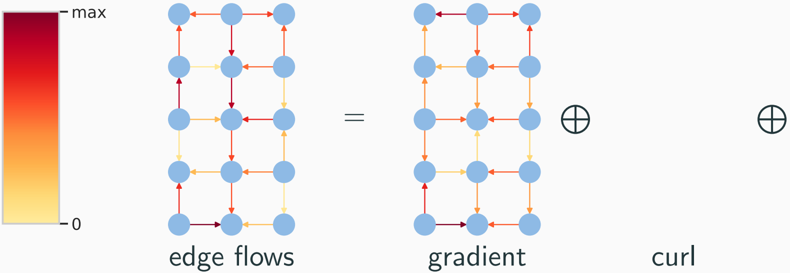

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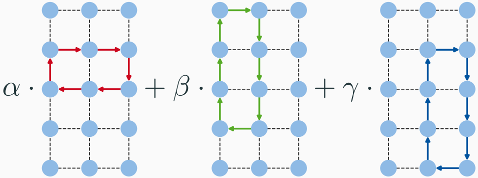

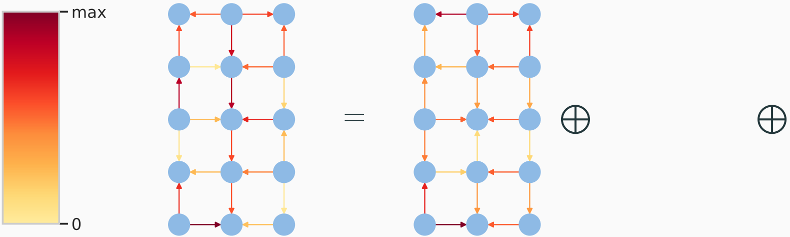

edge flows gradient curl

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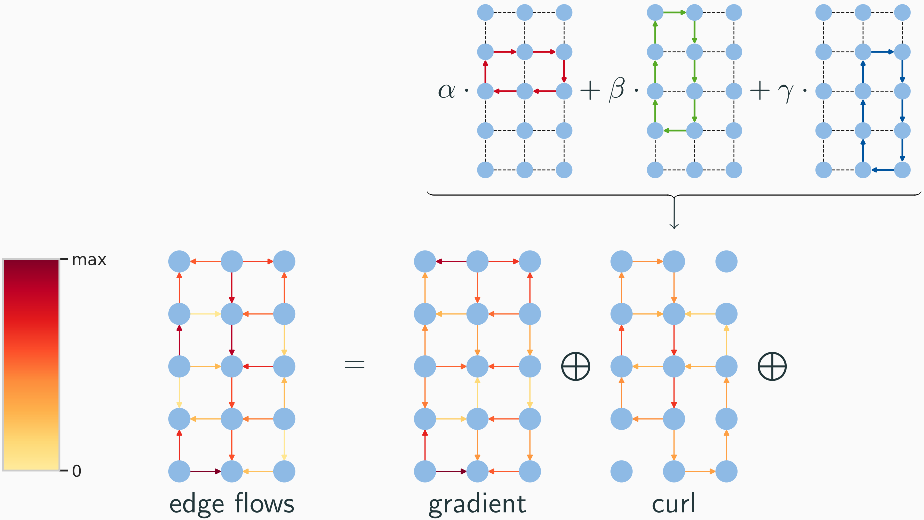

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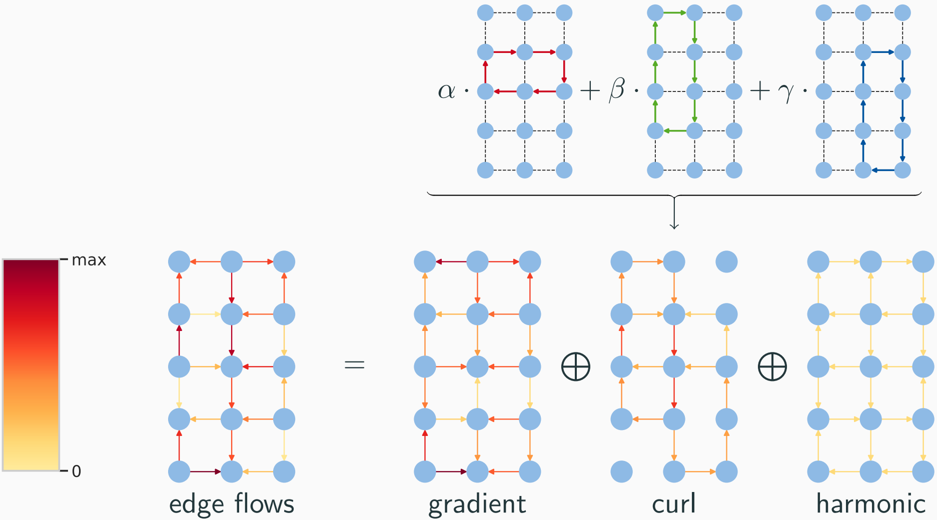

Problem Definition

Given

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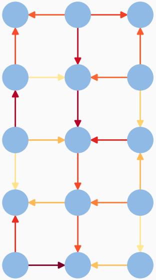
Given edge flows ,

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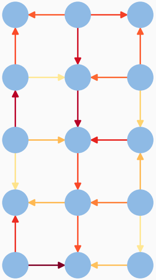
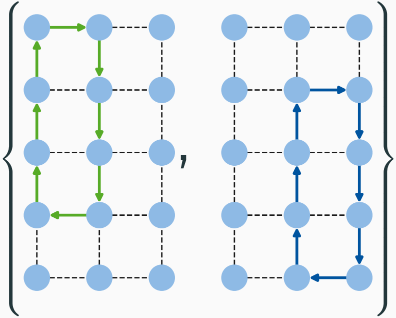
Given edge flows , find 2-cells 

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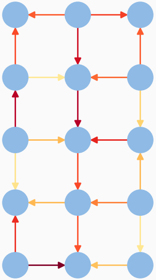
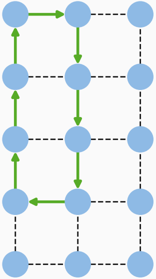
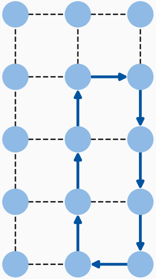
Given edge flows , find 2-cells { ,  } (cycles)

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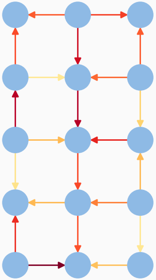

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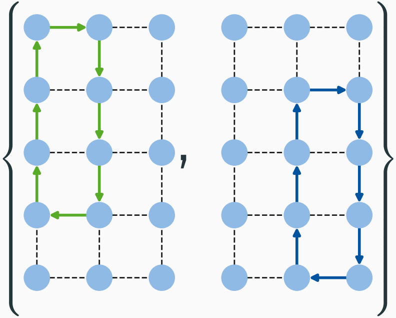

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that minimize the loss

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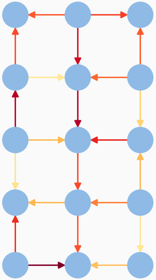
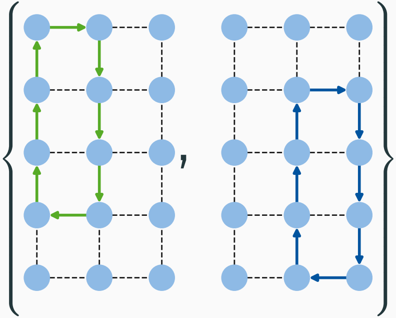
Given edge flows , find 2-cells  (cycles)

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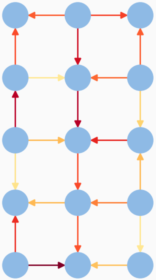
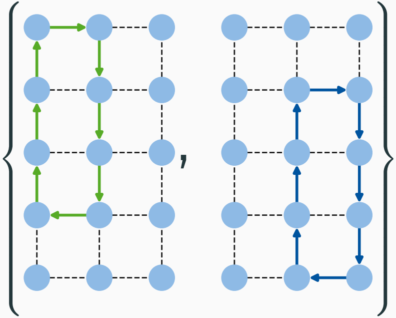
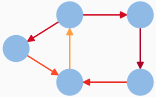
Given edge flows , find 2-cells  (cycles)

that minimize the loss (= least squares of harmonic flow)

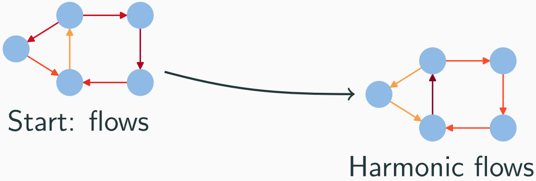
Note: The Decision Variant is **NP-Hard**.

Overview of the Algorithm



Start: flows

Overview of the Algorithm



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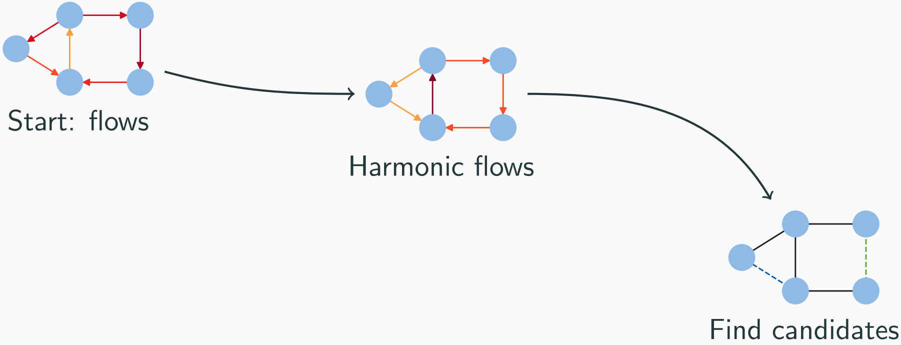

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Find candidates

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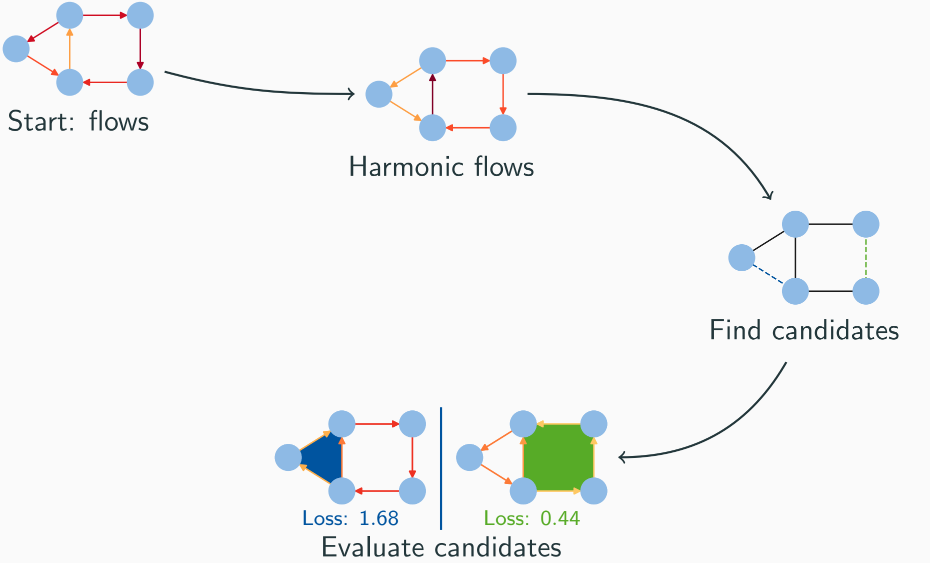

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Find candidates

Loss: 1.68

Loss: 0.44

Evaluate candidates

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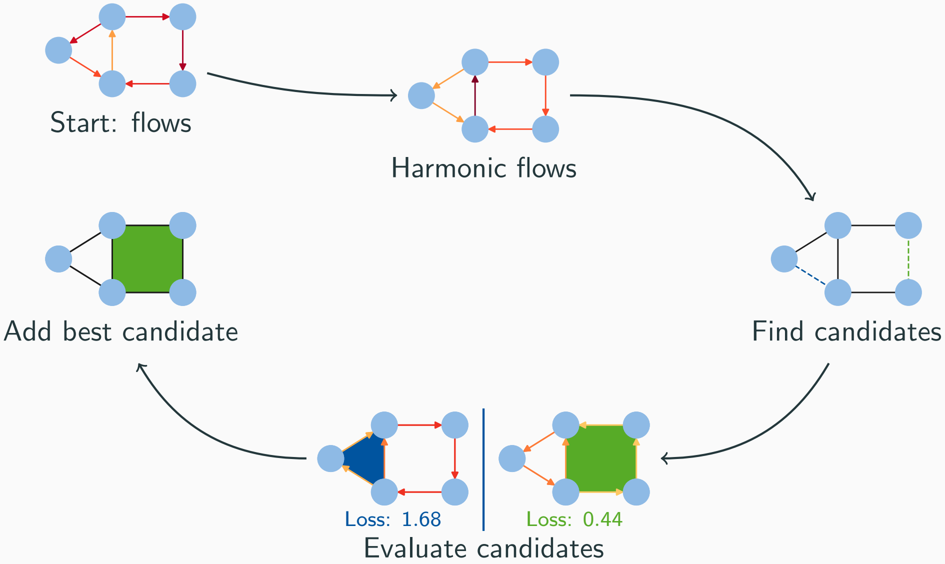

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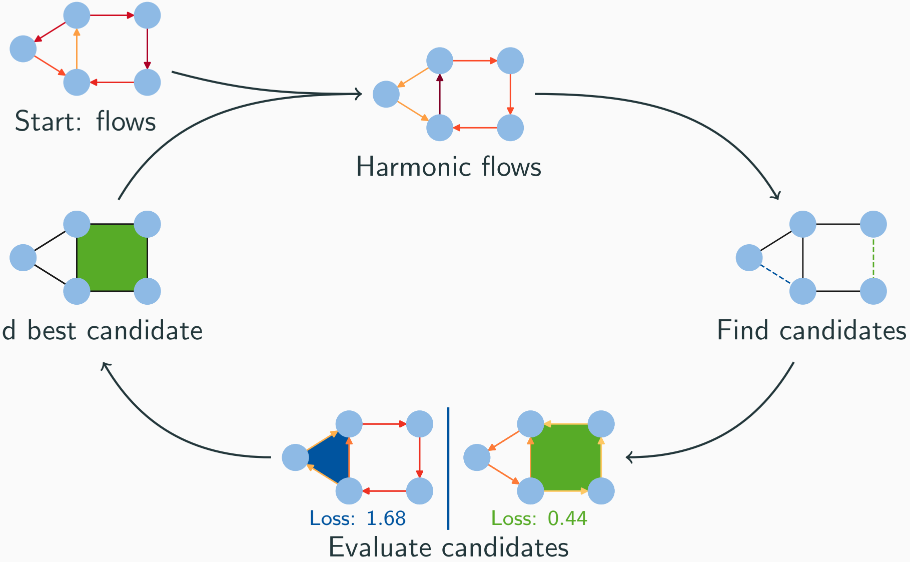

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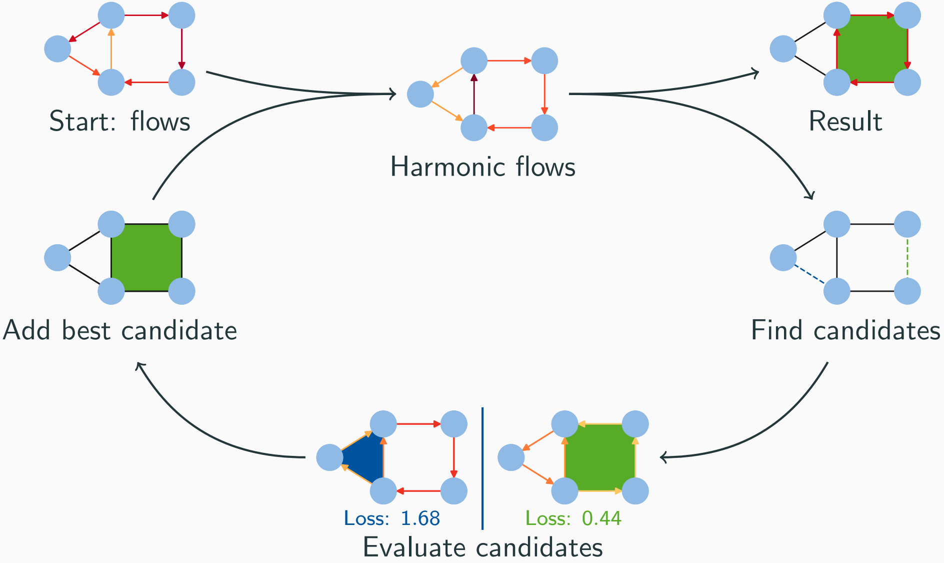
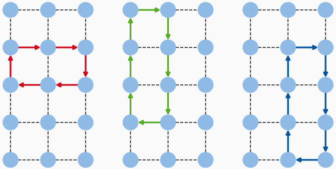


How can we get cell candidates?



predefined

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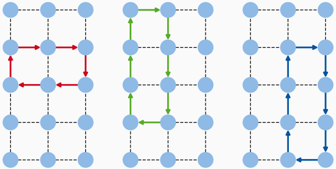

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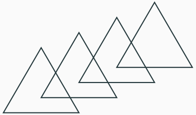

all triangles

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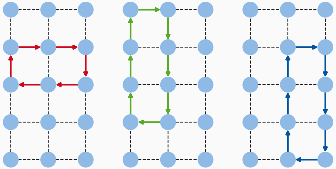

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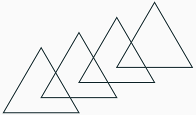

all triangles
(Simplicial Complex)

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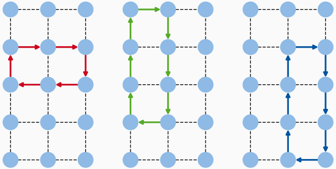

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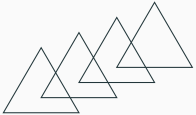

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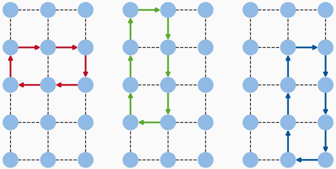

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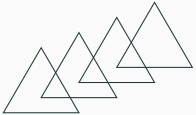

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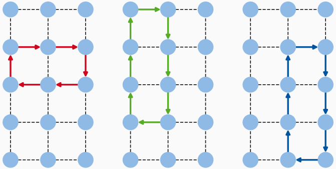

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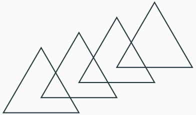

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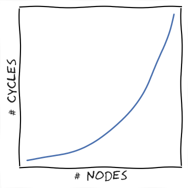

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Towards a heuristic for cell candidates

- Needs to consider “all” of the harmonic space

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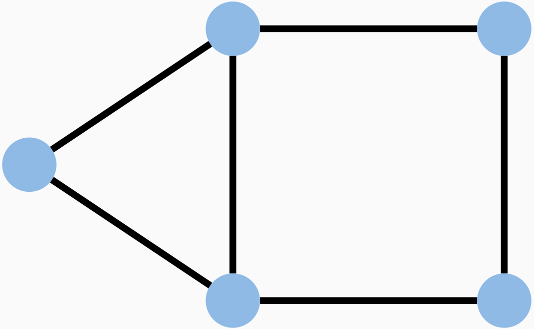
- Exponentially many cycles $\Rightarrow$ Must not consider all cycles

Towards a heuristic for cell candidates

- Needs to consider “all” of the harmonic space
- Exponentially many cycles ⇒ Must not consider all cycles
- Good start: Cycle basis
    - Spans entire harmonic space
    - Size linear in number of edges
    - Sparsity: Requires “good” cycles

Spanning Trees induce Cycle Bases



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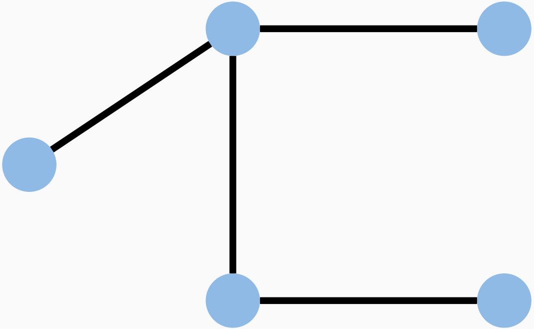

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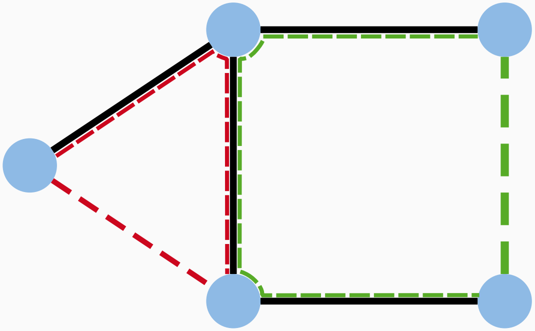

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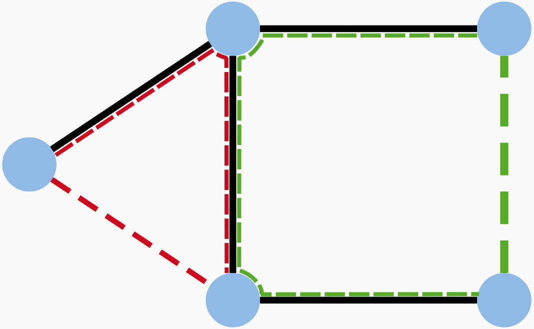
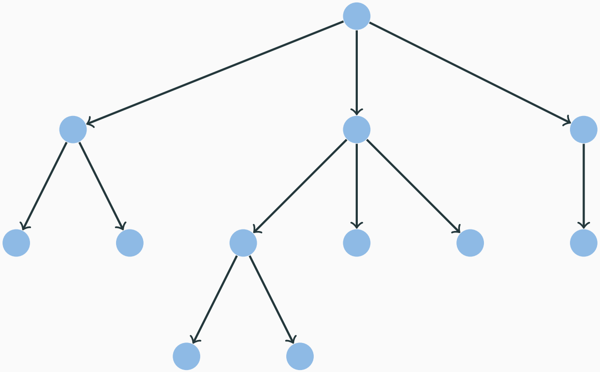


Complexity: exp(nodes) → nodes · edges

Utilizing Spanning Trees for efficiency



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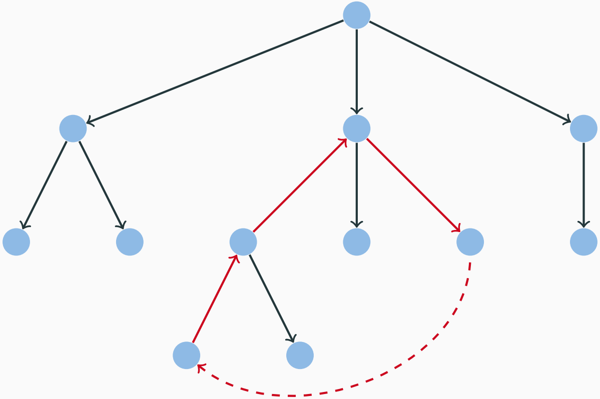

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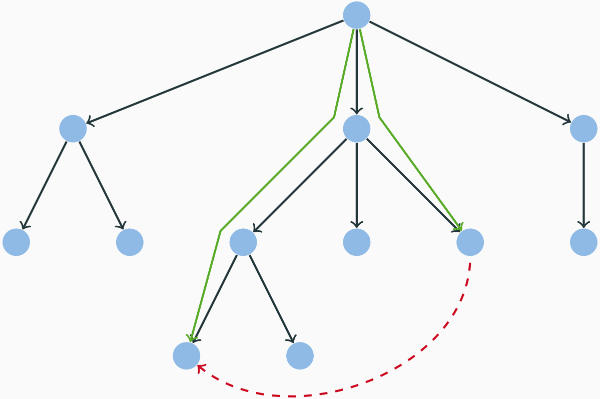

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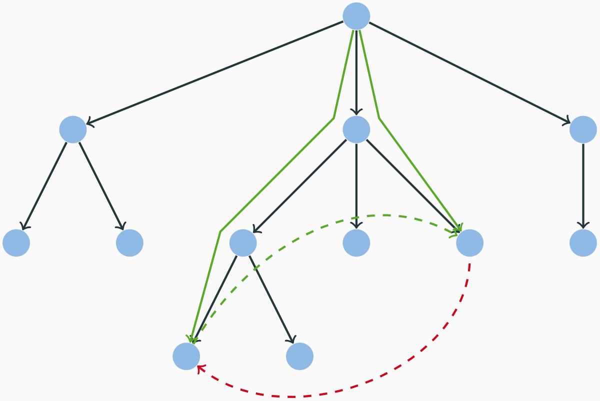

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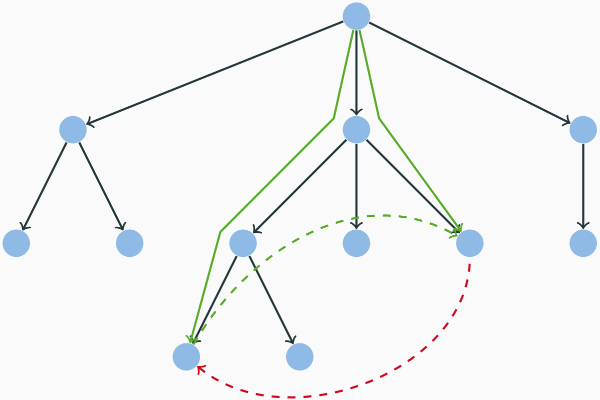
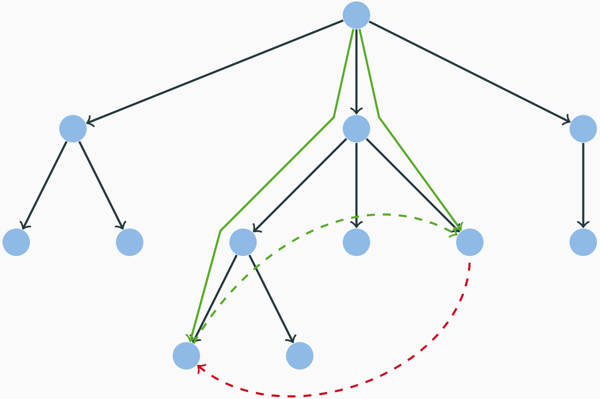


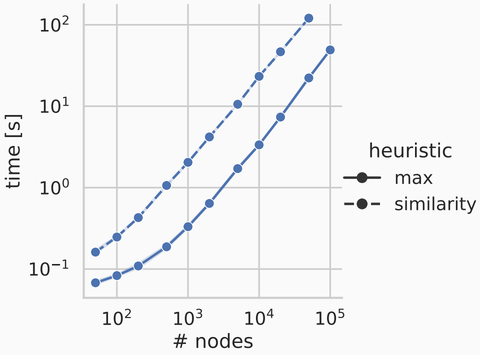
Complexity: nodes · edges → ∼edges + finding a spanning tree

Utilizing Spanning Trees for efficiency



Complexity: nodes · edges → ∼ edges + finding a spanning tree (edges log edges)

Results



Runtime on triangulation graph

[1] Transportation Networks for Research. `https://github.com/bstabler/TransportationNetworks`. Accessed 2023-08-18.

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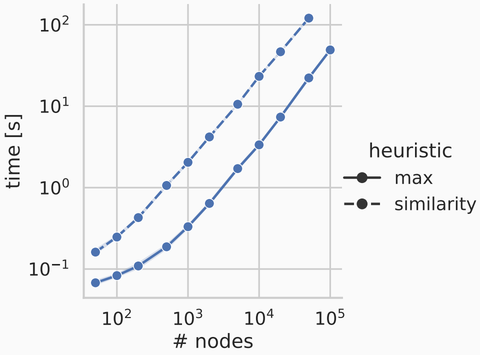

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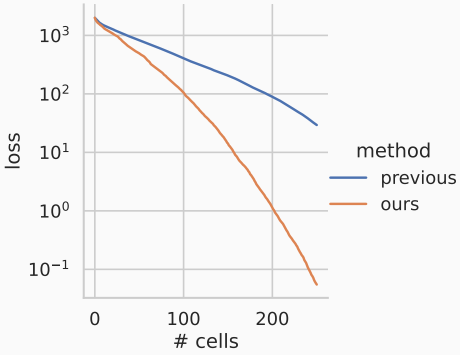


Traffic flow in Anaheim[1]

[1] Transportation Networks for Research. `https://github.com/bstabler/TransportationNetworks`. Accessed 2023-08-18.

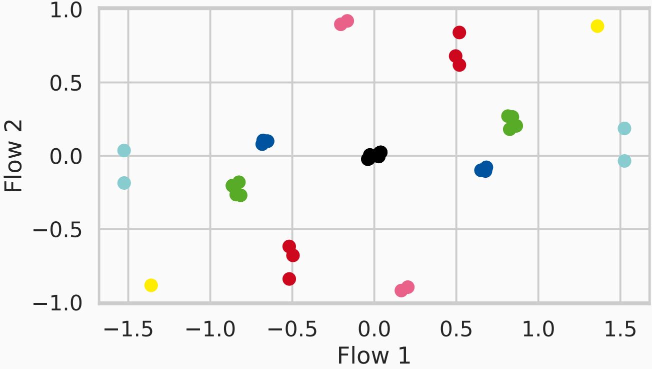
Conclusion

- Generalization of methodology for Topological Signal Processing
  - Cells: triangles → polygons
  - Candidates: pre-selected → heuristic selection
- Efficient approximation algorithm for NP-hard problem
  - Iterative greedy algorithm
  - Utilizing spanning trees for efficiency
  - Worked well in empirical experiments
- Future work:
  - Other downstream tasks / Adaptation to other tasks
  - Qualitative evaluation of inferred cells

Full Paper &
PyPI Package

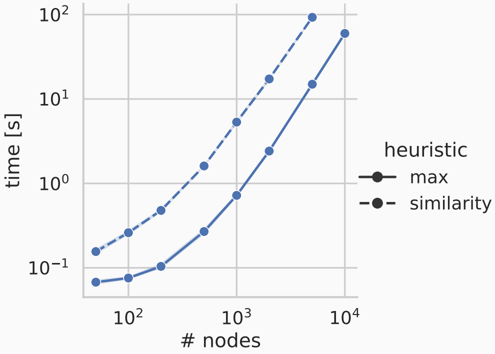
Heuristic: Finding Spanning Trees

1. Maximum spanning tree (by absolute flow)
2. Spanning tree of *similar* edges

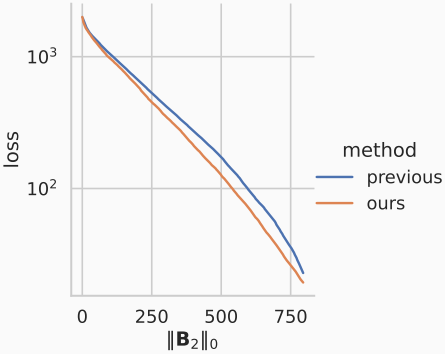


Edges clustered by flow value

Additional Plots



Runtime on smallworld graph



Traffic flow in Anaheim