Continuous Average Straightness in Spatial Graphs

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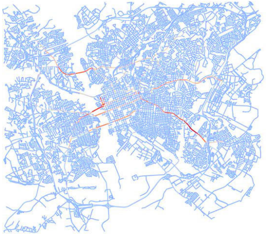
Outline

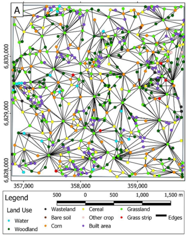
Spatial Graphs & Straightness
Spatial Graphs

- Spatial graphs [Bar11]: vertices have a spatial position, edges have a length
- Used to model systems in which spatial constraints affect the topology

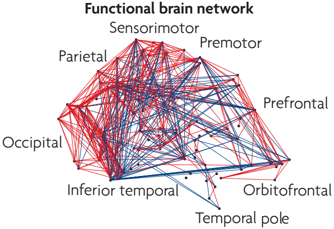


Road network
[Pun+22]

Electrical grid
[Sol+08]



Ecological network
[Bar+22]



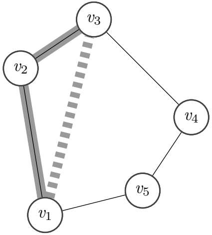
Cerebral network
[BS09]

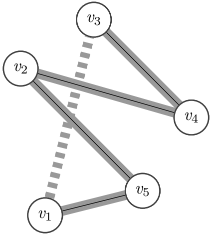
Spatial Graphs & Straightness
Straightness Measure

- Specific topological measures leverage spatial information [Bar11]
- Straightness [VLD05; PCL06]:

$$S(u, v) = \frac{d_E(u, v)}{d_G(u, v)}$$

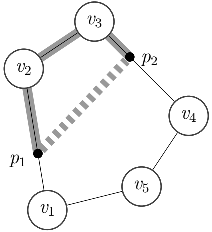
  - Characterizes a pair of vertices
  - Ratio of the Euclidean (d_E) to the graph (d_G) distances between them
    - ≈ 0 : very tortuous path
    - 1 : completely straight path
  - Interpretation: how efficient the graph is in providing the most direct path from one vertex to the other
  - Alternative names (sometimes for the reciprocal): Circuity [OM96], Directness [Hes97], Tortuosity [KT01], Route Factor [GN06], Detour Index [LJB13]...

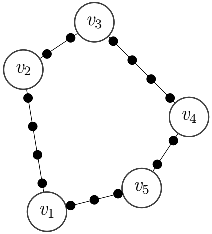




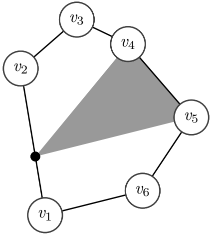
Spatial Graphs & Straightness
Averaging the Straightness

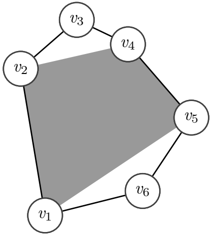
- Generally averaged in the literature:
  - Single vertex vs. rest of the graph [PCL06]
  - Subgraph: module [KT01], sample [Hes97; LE09]
  - Whole graph [KT01; VLD05; GN06; LJB13]
- Vertex-to-vertex vs. point-to-point routes
  - V2v: subway, airways…
  - P2p: soft transport transit
- Standard approach: discrete approximation
  1. Segment edges, adding extra vertices
  2. Compute Straightness for all vertex pairs
  3. Average these values





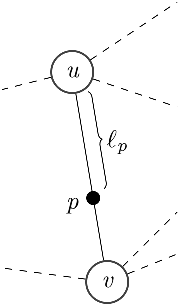
- Issues with the standard approach:
  - Computational cost (both time and memory)
  - Quality of the discrete approximation ?
- Proposed approach: continuous average
  - First integration: between a point and an edge
  - Second integration: between two edges
  - Sum over all pairs of edges



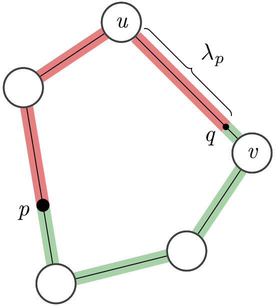


Reformulation of the Measure
Notations

- Which variable of integration?
  - → Relative position ℓ_p
  - = Position of p relative to the first vertex u of its edge (u, v)
  - ≠ By opposition to its absolute position (x_p, y_p)
- Constants:
  - Absolute position of vertices
  - Graph distance between vertices
- How to reformulate S as a function of ℓ?
  - → Straightforward for the Euclidean distance
  - → More complicated for the graph distance



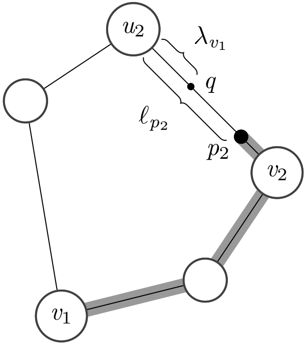
Relative position

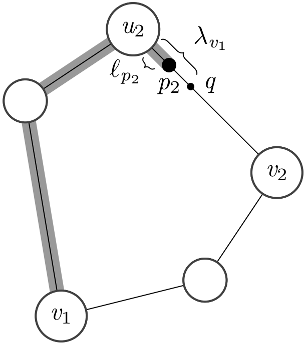


Break-even point and distance

Reformulation of the Measure
Vertex-to-Point Graph Distance

- Graph distance between vertex v_1 and point p_2 :
  → Graph distance between v_1 and u_2 or v_2 + segment length on (u_2, v_2)
- Which path is the shortest?
  → Depends on the position of p_2 w.r.t. the break-even point q

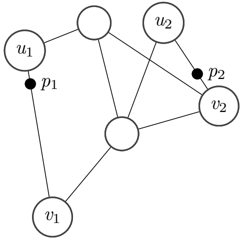


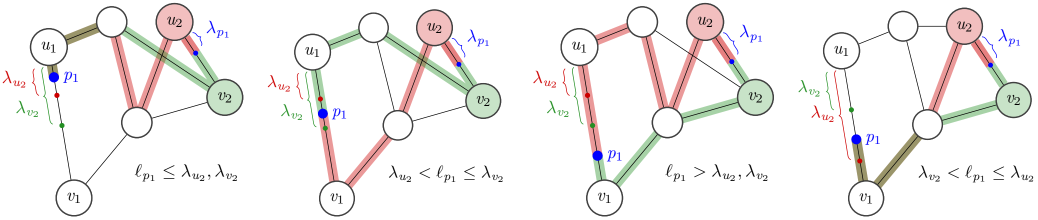


Reformulation of the Measure
Point-to-Point Graph Distance

- Starting from a non-vertex point p_1
- We first want λ_{p_1} (BEP of p_1):
  - Depends on the position of p_1 relative to λ_{v_2} and λ_{u_2}

→ Must consider 4 situations

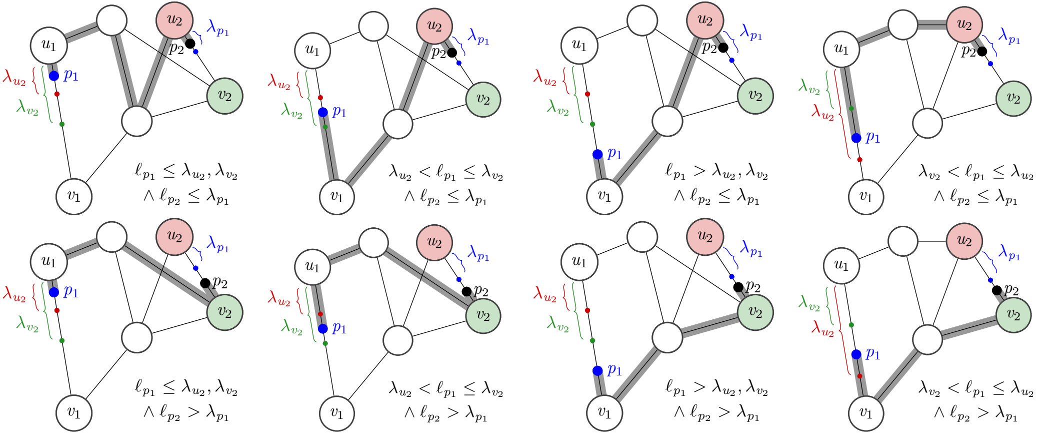




Reformulation of the Measure
Point-to-Point Graph Distance

- Next, leverage λ_{p_1} to characterize the position of p_2

→ Total of 8 cases to consider when computing the graph distance



Continuous Average Straightness
Point-Oriented Variants

Straightness expressed as $S(p_1, p_2) = f(\ell_{p_1}, \ell_{p_2})$

Average Straightness between a point and an edge

Let $p_1 \in P$ be a point and $(u_2, v_2) \in E$ an edge. The average Straightness between them is

$$S_{u_2 v_2}(p_1) = \frac{1}{d_E(u_2, v_2)} \int_{(u_2,v_2)} f(\ell_{p_1}, \ell_{p_2}) \mathrm{d}\ell_{p_2}. \quad (1)$$

Average Straightness between a point and the graph

Let G be a graph and $p_1 \in P$ one of its points. The average Straightness between them is

$$S_G(p_1) = \frac{1}{\sum\limits_{(u_2,v_2)} d_E(u_2, v_2)} \sum_{(u_2,v_2)} \int_{(u_2,v_2)} f(\ell_{p_1}, \ell_{p_2}) \mathrm{d}\ell_{p_2}. \quad (2)$$

Continuous Average Straightness
Edge-Oriented Variants

Average Straightness between two edges

Let (u_1, v_1) and $(u_2, v_2) \in E$ be two edges. The average Straightness between them is

$$S_{u_2 v_2}(u_1, v_1) = \frac{1}{d_E(u_1, v_1) d_E(u_2, v_2)} \int_{(u_1,v_1)} \int_{(u_2,v_2)} f(\ell_{p_1}, \ell_{p_2}) \mathrm{d}\ell_{p_1} \mathrm{d}\ell_{p_2}. \quad (3)$$

Average Straightness between an edge and the graph

Let G be a graph and $(u_1, v_1) \in E$ one of its edges. The average Straightness between them is

$$S_G(u_1, v_1) = \frac{1}{\sum_{(u_2,v_2)} d_E(u_1, v_1) d_E(u_2, v_2)} \sum_{(u_2,v_2)} \int_{(u_1,v_1)} \int_{(u_2,v_2)} f(\ell_{p_1}, \ell_{p_2}) \mathrm{d}\ell_{p_1} \mathrm{d}\ell_{p_2}. \quad (4)$$

Average Straightness for the whole graph

The Straightness averaged over the whole graph G is

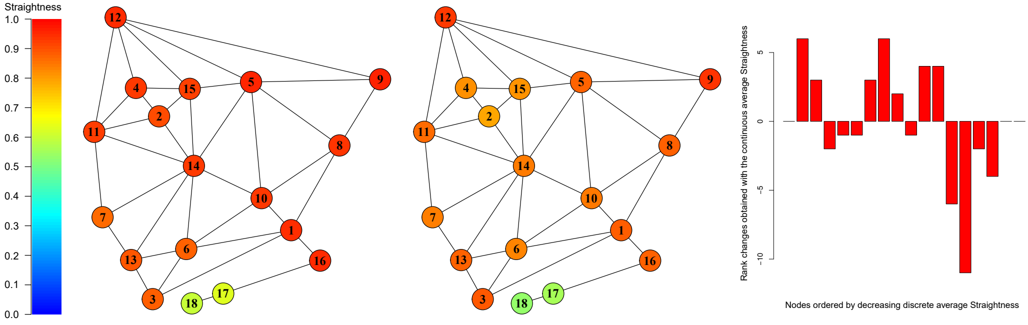
$$S_G = \frac{1}{\sum_{(u_1,v_1)} \sum_{(u_2,v_2)} d_E(u_1, v_1) d_E(u_2, v_2)} \sum_{(u_1,v_1)} \sum_{(u_2,v_2)} \int_{(u_1,v_1)} \int_{(u_2,v_2)} f(\ell_{p_1}, \ell_{p_2}) \mathrm{d}\ell_{p_1} \mathrm{d}\ell_{p_2}. \qquad (5)$$

Computational notes:

- Closed form of the first integral → hard-coded
- Not for the second one → numerical approximation

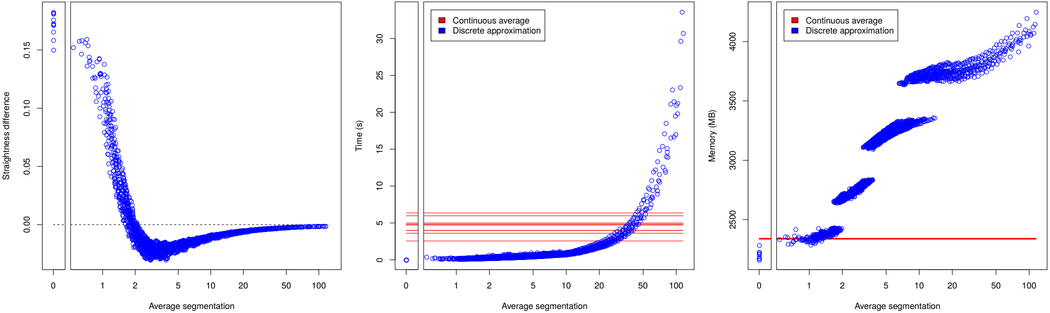
Empirical Assessment
Toy Networks

- Data: regular graphs and small manually designed graphs
- Qualitative comparison between the discrete version and $S_G(u)$, i.e. average Straightness between a vertex and its graph:



Empirical Assessment
Random Planar Graphs

- Data: benchmark of random planar graphs
- Considered aspects:
  - Difference between the continuous average S and its discrete approximation
  - Computational time
  - Memory usage
- For S_G, i.e. average Straightness over the whole graph:



Empirical Assessment
Random Planar Graphs

- Data: benchmark of random planar graphs
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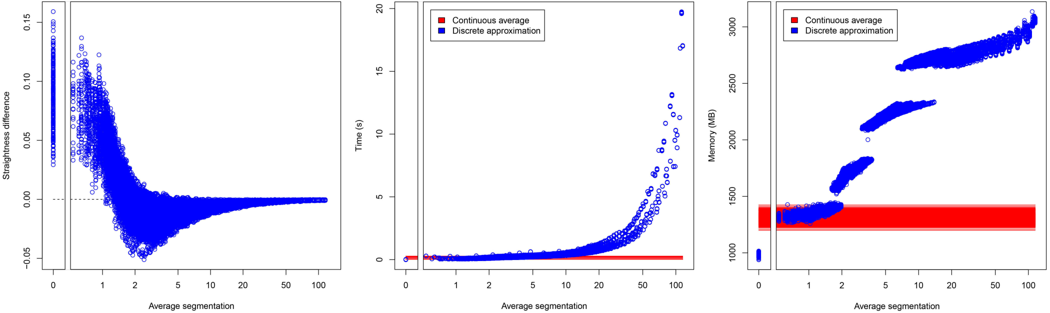
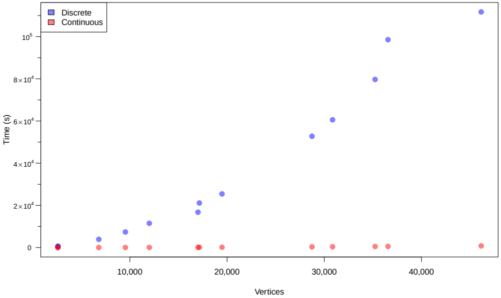
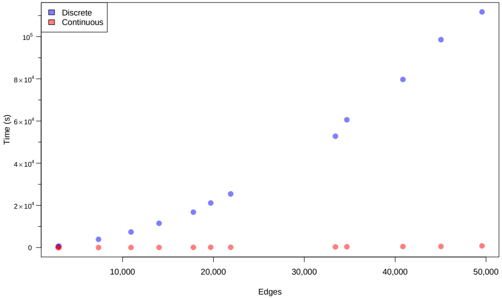
- For $S_G(u)$, i.e. average Straightness between a vertex and its graph:



Empirical Assessment
Real-World Road Networks

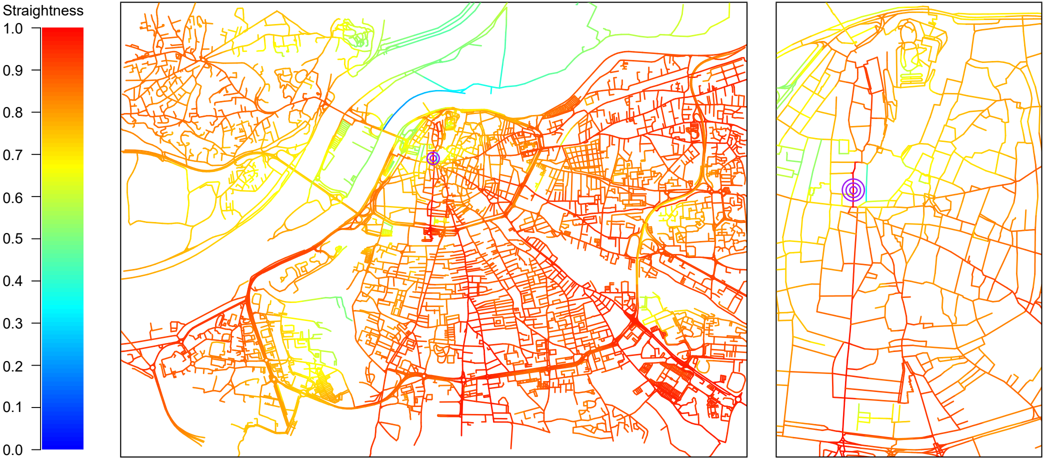
- Data: real-world urban networks retrieved from OpenStreetMap
- Computation time vs. network size (# vertices and # edges)
- Discrete version and $S_G(u)$, i.e. average Straightness between a vertex and its graph:





Empirical Assessment
Real-World Road Networks

Example : Avignon, average Straightness between the city hall and each edge ($S_{u_2 v_2}(p_1)$)



Conclusion & Perspectives

- Takeaways: continuous approach to average the Straightness
  - → Faster, requires less memory
  - → More reliable Straightness value
  - → Definition of 5 distinct average variants of the measure
- Resources
  - Paper: V. Labatut. "Continuous Average Straightness in Spatial Graphs". In: *Journal of Complex Networks* 6.2 (2018), pp. 269–296. DOI: `10.1093/comnet/cnx033`
  - Source code: `https://github.com/CompNet/SpatialMeasures`
- Perspectives
  - Apply the same principle to other spatial measures
  - Generation of networks optimizing straightness (and other criteria)

Questions?

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Section 5

Additional Material

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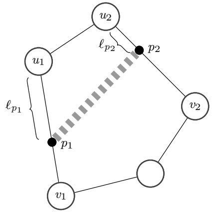
Euclidean Distance Between Two Points

Lemma 3.1 (Euclidean distance between two points)

Let $p_1 \in P$ *be a point lying on an edge* $(u_1, v_1) \in E$ *at a distance* ℓ_{p_1} *from* u_1*, and* $p_2 \in P$ *be a point lying on an edge* $(u_2, v_2) \in E$ *at a distance* ℓ_{p_2} *from* u_2*.*
The Euclidean distance between p_1 *and* p_2 *can be written in terms of* ℓ_{p_1} *and* ℓ_{p_2}*, as:*

$$d_E(p_1, p_2) = \left(\left(x_{u_2} + \frac{\ell_{p_2}}{d_E(u_2, v_2)}(x_{v_2} - x_{u_2}) - x_{u_1} - \frac{\ell_{p_1}}{d_E(u_1, v_1)}(x_{v_1} - x_{u_1}) \right)^2 + \left(y_{u_2} + \frac{\ell_{p_2}}{d_E(u_2, v_2)}(y_{v_2} - y_{u_2}) - y_{u_1} - \frac{\ell_{p_1}}{d_E(u_1, v_1)}(y_{v_1} - y_{u_1}) \right)^2 \right)^{1/2}. \quad (6)$$

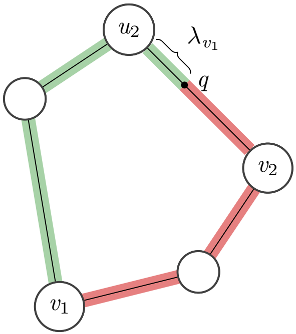


Break-Even Distance for a Vertex

Lemma 3.2 (Break-even distance for a vertex)

Let $v_1 \in V$ be a vertex and $(u_2, v_2) \in E$ an edge such that $v_1 \neq v_2$ and $v_1 \neq u_2$. The break-even distance of (u_2, v_2) for the vertex v_1, noted λ_{v_1} is:

$$\lambda_r = \frac{d_G(v_1, v_2) - d_G(v_1, u_2) + d_E(u_2, v_2)}{2}. \quad (7)$$

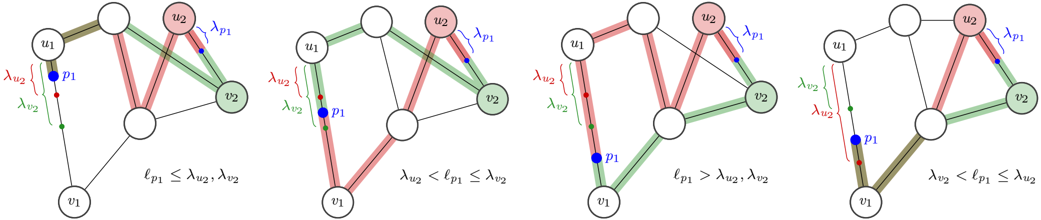


Break-Even Distance for a Point

Lemma 3.4 (Break-even distance for a point)

Let $p_1 \in P$ be a point lying on an edge $(u_1, v_1) \in E$, at a distance ℓ_{p_1} from u_1, and $(u_2, v_2) \in E$ be a distinct edge (though they can have one common end-vertex). Let the break-even distances of (u_1, v_1) for u_2 and v_2 be λ_{u_2} and λ_{v_2}, respectively. The break-even distance λ_{p_1} of (u_2, v_2) for p_1 is:

$$\lambda_{p_1} = \begin{cases} \lambda_{p_1}^{(1)} = \left(d_G(u_1, v_2) - d_G(u_1, u_2) + d_E(u_2, v_2)\right)/2, & \text{if } \ell_{p_1} \leq \lambda_{u_2}, \lambda_{v_2}, \\ \lambda_{p_1}^{(2)} = \left(d_G(v_1, v_2) - d_G(u_1, u_2) + d_E(u_1, v_1) + d_E(u_2, v_2) - 2\ell_{p_1}\right)/2, & \text{if } \lambda_{v_2} < \ell_{p_1} \leq \lambda_{u_2}, \\ \lambda_{p_1}^{(3)} = \left(d_G(u_1, v_2) - d_G(v_1, u_2) - d_E(u_1, v_1) + d_E(u_2, v_2) + 2\ell_{p_1}\right)/2, & \text{if } \lambda_{u_2} < \ell_{p_1} \leq \lambda_{v_2}, \\ \lambda_{p_1}^{(4)} = \left(d_G(v_1, v_2) - d_G(v_1, u_2) + d_E(u_2, v_2)\right)/2, & \text{if } \ell_{p_1} > \lambda_{u_2}, \lambda_{v_2}. \end{cases} \quad (8)$$



Empirical Assessment
Real-World Road Networks

City	n	m	δ	Processing time			
				Discrete		Continuous	
Abidjan	2,577	2,908	$8.76 \cdot 10^{-4}$	490.12	(0.02)	4.00	(0.08)
Karlskrona	2,619	2,999	$8.75 \cdot 10^{-4}$	672.36	(0.02)	4.11	(0.06)
Soustons	6,803	7,351	$3.18 \cdot 10^{-4}$	3,889.20	(0.10)	18.13	(0.10)
Maastricht	9,539	10,929	$2.40 \cdot 10^{-4}$	7,379.10	(0.08)	35.84	(0.18)
Trois-Rivières	12,001	14,014	$1.95 \cdot 10^{-4}$	11,504.23	(0.12)	55.54	(0.26)
Alice Springs	17,011	17,790	$1.23 \cdot 10^{-4}$	16,765.85	(0.17)	93.38	(0.56)
Sfax	17,152	19,702	$1.34 \cdot 10^{-4}$	21,122.19	(0.14)	131.18	(0.42)
Avignon	19,481	21,898	$1.15 \cdot 10^{-4}$	25,438.31	(0.20)	136.30	(0.77)
Liverpool	28,739	33,424	$8.09 \cdot 10^{-5}$	52,773.02	(0.20)	313.55	(3.32)
Ljubljana	30,854	34,684	$7.29 \cdot 10^{-5}$	60,572.10	(0.31)	369.35	(26.91)
Lisbon	35,231	40,853	$6.58 \cdot 10^{-5}$	79,702.34	(0.44)	495.99	(9.46)
Dakar	36,561	45,041	$6.74 \cdot 10^{-5}$	98,568.74	(0.40)	530.90	(7.62)
Hong Kong	46,145	49,559	$4.65 \cdot 10^{-5}$	111,724.71	(0.36)	790.45	(8.67)

Section 5

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