Kempe equivalence of colourings

Marthe Bonamy Nicolas Bousquet
Carl Feghali Matthew Johnson

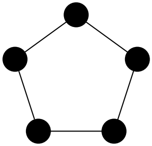
June 3rd, 2016

LaBRI

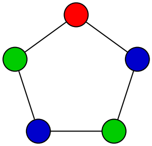
ÉCOLE CENTRALE LYON

Durham University

Graph



Graph coloring



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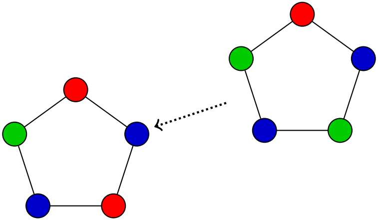

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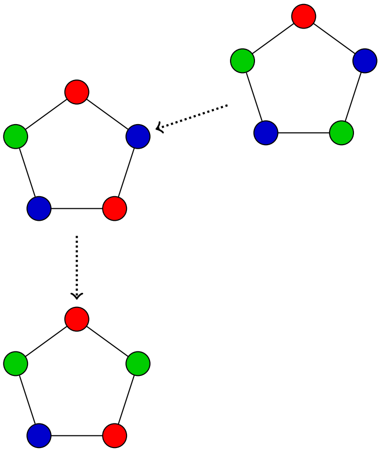

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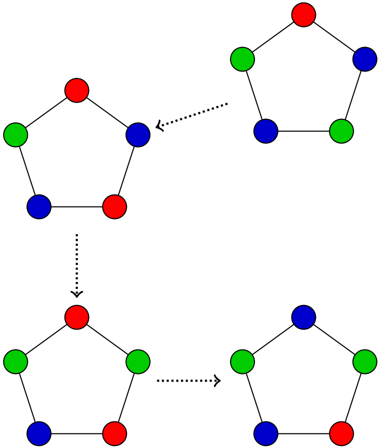

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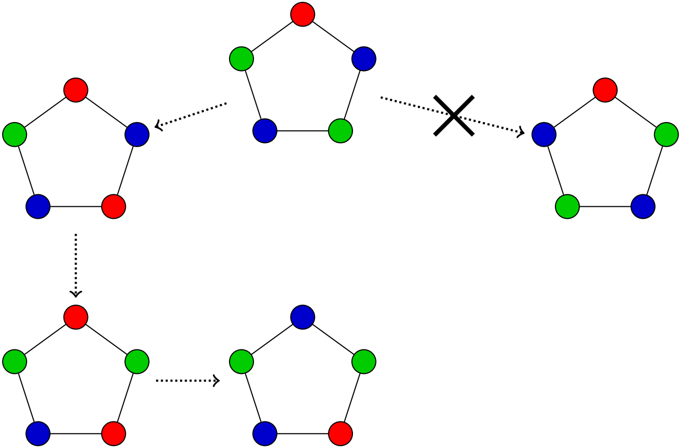
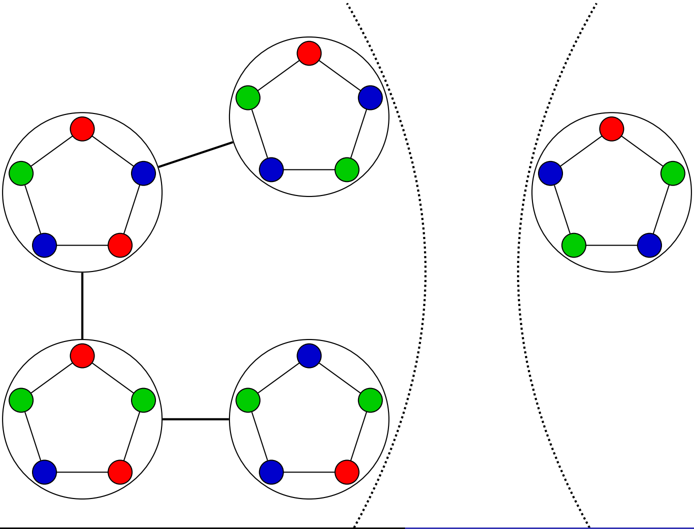
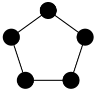


Graph recoloring ⇒ Reconfiguration graph

Solutions // Nodes. Most similar solutions // Neighbors.



Necessary condition

 is not 3-recolorable.

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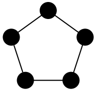
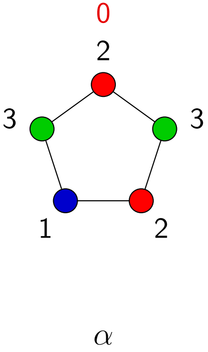
 is not 3-recolorable.



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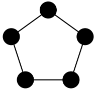

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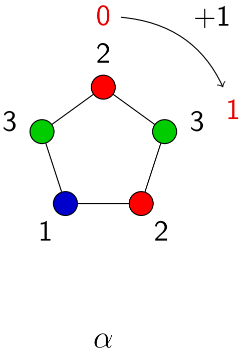

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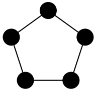
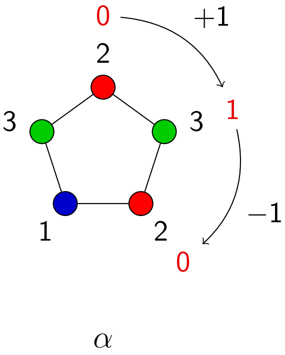
 is not 3-recolorable.



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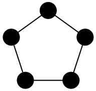
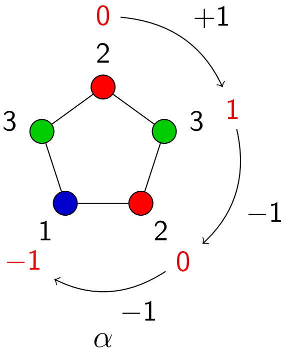
 is not 3-recolorable.



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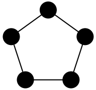
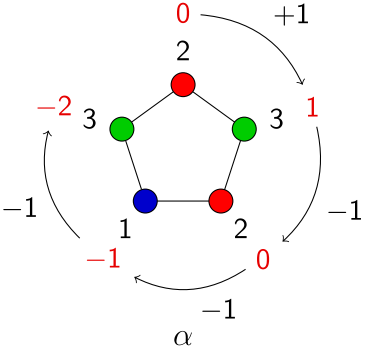
 is not 3-recolorable.



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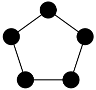
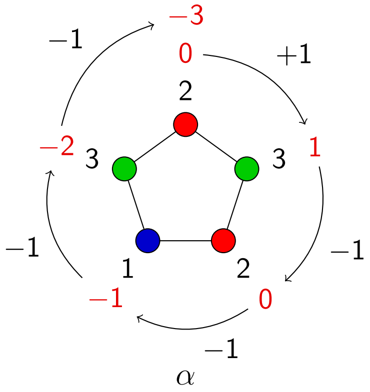
 is not 3-recolorable.



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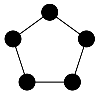
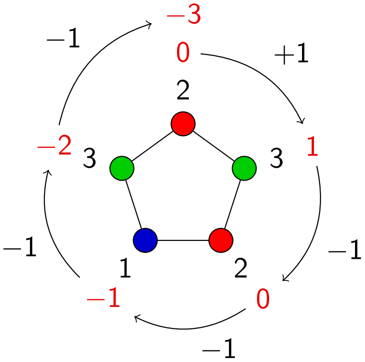
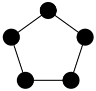
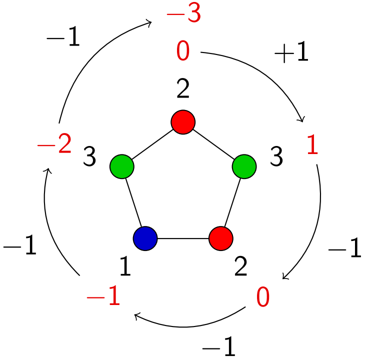
 is not 3-recolorable.



α

$$w(\alpha, C_5) = -3$$

 is not 3-recolorable.



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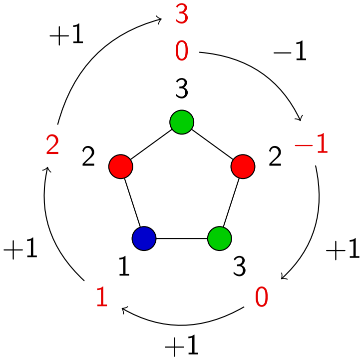
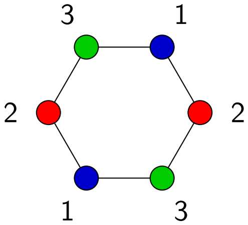


β

$w(\beta, C_5) = 3$

Another necessary condition



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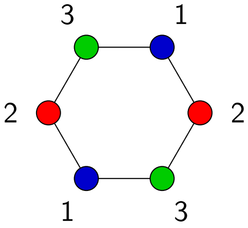

Frozen cycle

Two sufficient conditions

Theorem (Wrochna '14)

Let α, β be 3-colorings of a graph G. If

- $w(C, \alpha) = w(C, \beta)$ *for every cycle* C *in* G, *and*
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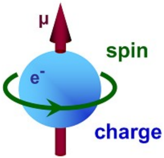
$\rightsquigarrow$ 3-coloring β as $\beta(w) = \alpha(w) + 1 \pmod 3$ for every vertex w.
$\Rightarrow$ $G - uv$ can be recolored from α to β. $\square$

Anti-ferromagnetic Potts Model



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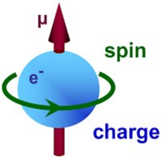

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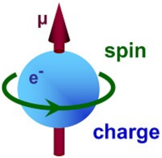

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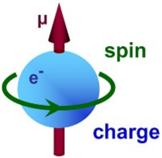

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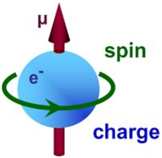

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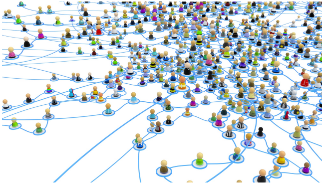
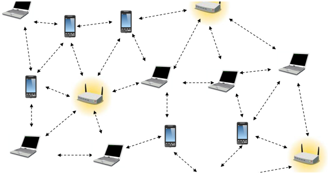
The physicists want to:

- Find the mixing time of Markov chains on Glauber dynamics.
  We need to recolor only one vertex at a time.
- Generate all the possible states of a Glauber dynamics.
  We have no constraint on the method.

Limit of the recoloring model

- In many applications, colors are interchangeable.





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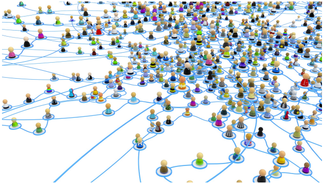
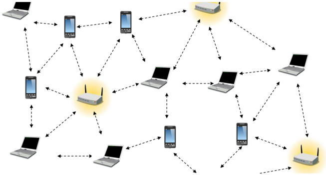
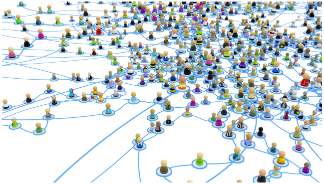
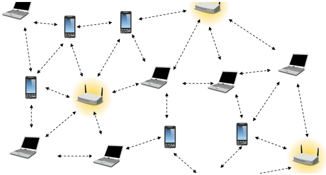




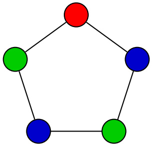
Limit of the recoloring model

- In many applications, colors are interchangeable.
- More actions may be available.
- Which type of actions ensures that the reconfiguration graph is connected?





Kempe chains



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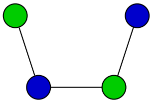

Let a, b be two colors.

- A connected component of the graph induced by the vertices colored by a or b is a Kempe chain.

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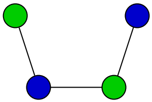

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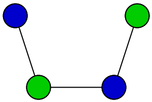
- Permuting the colors of a Kempe chain is a Kempe change.



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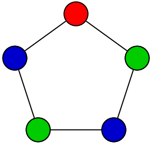

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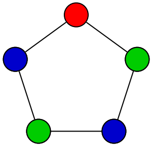

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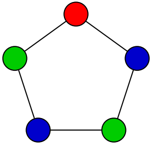
$\Rightarrow$ Kempe changes generalize single vertex recolorings.

Kempe equivalence



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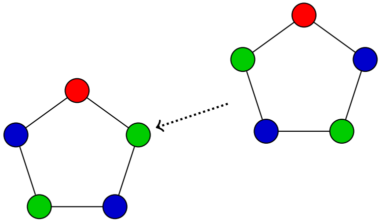

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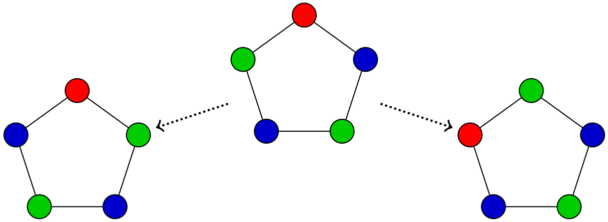

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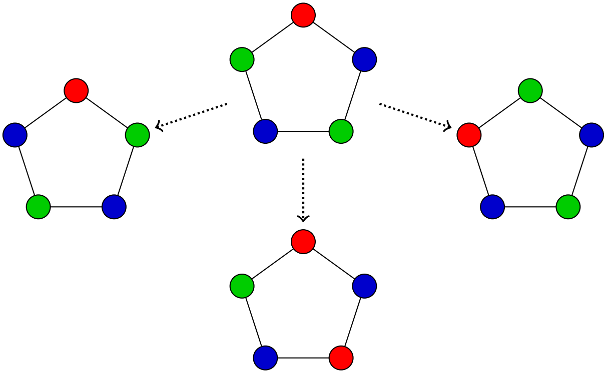

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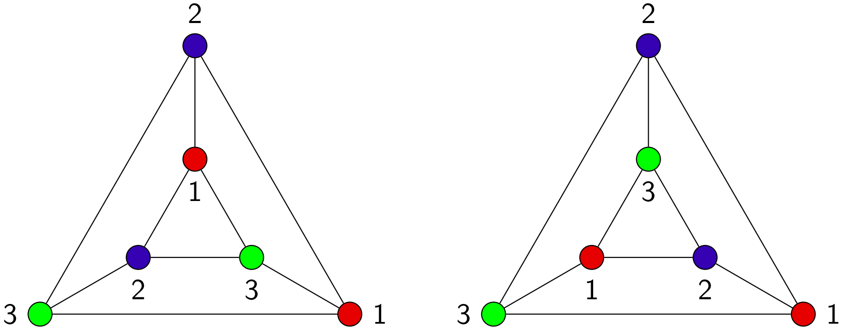
The conjecture is false! (van den Heuvel '13)



(3-prism)

Results (2)

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Theorem (B., Bousquet, Feghali, Johnson '15)

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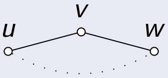
Main observation

Lemma (Meyniel, Las Vergnas '01)

In a 3-connected *graph with* 

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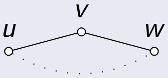
In a 3-connected graph with 

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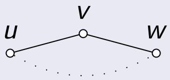
Sketch:

- Identify u and w.
- The resulting graph is connected and $(\Delta - 1)$-degenerate.
- Δ-colorings of a $(\Delta - 1)$-degenerate graph are equivalent.

Main observation

Lemma (Meyniel, Las Vergnas '01)

In a 3-connected graph with 

, all the colorings where u and w are colored alike are Kempe equivalent.

Sketch:

- Identify u and w.
- The resulting graph is connected and $(\Delta - 1)$-degenerate.
- Δ-colorings of a $(\Delta - 1)$-degenerate graph are equivalent.

Consequence: If every coloring is equivalent to a coloring where u and w are colored alike, all the colorings are Kempe equivalent.

$$\begin{array}{ccc} \Delta\text{-coloring } \alpha & & \Delta\text{-coloring } \beta \\ \Downarrow & & \Uparrow \\ \Delta\text{-col. } \alpha' \text{ where } \alpha'(u) = \alpha'(w) & \Rightarrow & \Delta\text{-col. } \beta' \text{ where } \beta'(u) = \beta'(w) \end{array}$$

Sketch of the main result

Theorem (B., Bousquet, Feghali, Johnson '15)

All the colorings of a connected k-regular graph with $k \geq 4$ are Kempe equivalent.

By contradiction: let G be a minimal k-regular graph with ≥ 2 Kempe classes.

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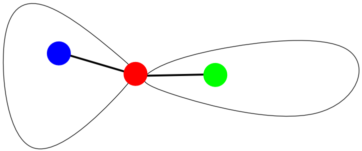

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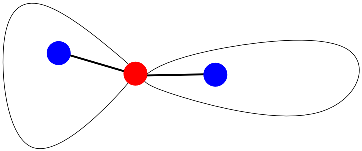

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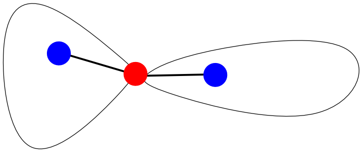
- If G does not have diameter at least 3 $\Rightarrow$ contradiction.

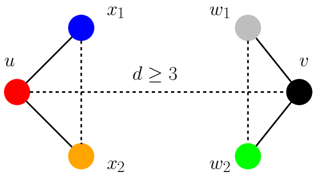
$\Rightarrow$ G is 3-connected of diameter ≥ 3.



Sketch (2)

So G is 3-connected of diameter ≥ 3.

- Let u, v at distance ≥ 3.
- Let w_1, w_2 in $N(u)$ s.t. $(w_1, w_2) \notin E$.
- Let x_1, x_2 in $N(v)$ s.t. $(x_1, x_2) \notin E$.



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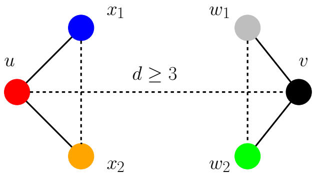

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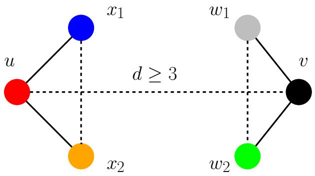

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for any Δ-coloring, two neighbors of v are colored the same

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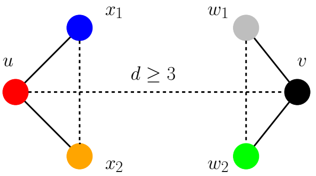

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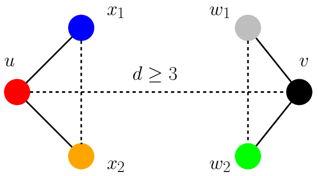

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Sketch:

Δ-coloring α $\downarrow$ Δ-col. α s.t. $\alpha(x_1) = \alpha(x_2)$ $\searrow$ Δ col. γ s.t. $\gamma(w_1) = \gamma(w_2)$, $\gamma(x_1) = \gamma(x_2)$ $\rightarrow$ Δ col. γ' s.t. $\gamma'(w_1) = \gamma'(w_2)$, $\gamma'(x_3) = \gamma'(x_4)$ $\nearrow$ Δ-col. β s.t. $\beta(x_3) = \beta(x_4)$ $\uparrow$ Δ-coloring β

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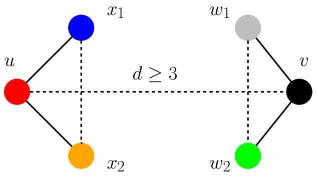

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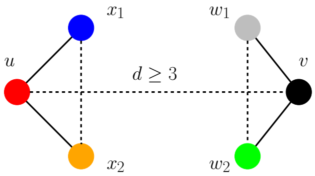

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- Erdős et al '79: A connected graph is degree-choosable unless every 2-connected component is a clique or an odd cycle.

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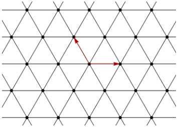

Consequence in physics: Close the study of the Wang-Swendsen-Koteký algorithm for Glauber dynamics on triangular lattices.

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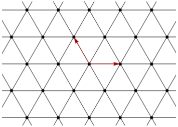

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Thanks for your attention!