Induced subdigraphs of digraphs with large chromatic number

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Outline

Clique number and chromatic number

Let G be a graph.

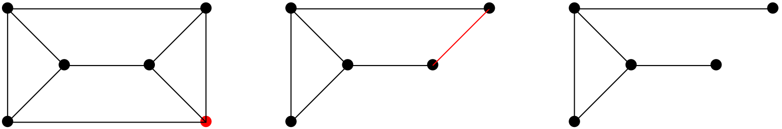
- $\chi(G)$ denote the chromatic number of G, that is the smallest integer k such that one can color G with k colors in such a way that no two adjacent vertices receive the same color.

- $\omega(G)$ denote the clique number of G, that is the size of a maximum clique in G (that is a maximum set of pairwise adjacent vertices).

$$\omega(G) \leq \chi(G)$$

Subgraphs of graphs with large chromatic number

Subgraph : subgraph obtained by deleting edges or vertices.



What can we say about the subgraphs of a graph with large chromatic number ?

More precisely : if you fix a graph H and you give me a graph G with sufficiently large chromatic number, can I ensure you that G contains H as a subgraph ?

Example : K_n contains all graphs of order n as subgraphs.

Subgraphs of graphs with large chromatic number

Theorem (Erdős, 1959) : There exists graphs with arbitrarily large chromatic number and girth.

Girth $=$ length of a smallest cycle.

Subgraphs of graphs with large chromatic number

Theorem (Erdős, 1959) : There exists graphs with arbitrarily large chromatic number and girth.

Girth = length of a smallest cycle.

If a graph H has a cycle, then there exists a graph G with no copy of H and arbitrarily large chromatic number.

Theorem : If $\chi(G) \geq k$, then G contains all trees T of order k as a subgraph.

If all vertices of G have degree at least $k-1$, then G contains all trees of order k as a subgraph.

Proof : by induction on k.

Outline

The oriented version

The chromatic number (resp. the clique number) of a digraph is the chromatic number (resp. the clique number) of its underlying graph.

What can we say about the subdigraphs of a digraph with large chromatic number ?

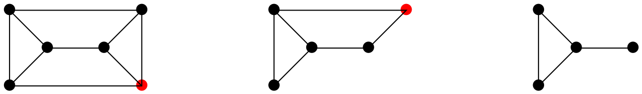
Theorem (Burr, 1980) : If $\chi(D) \geq (k-1)^2$, then D contains all trees of order k as subdigraphs.
Proof : by induction on k.

Conjecture (Burr, 1980) : if $\chi(D) \geq 2k-2$, then D contains all trees of order k as subdigraphs.

Outline

Large chromatic number and induced subgraphs

Induce subgraph : subgraph obtained by deleting vertices.



What can we say about the induced subgraphs of a graph with large chromatic number ?

Let H be a graph. The class of graphs that do not contain H as an induced subgraph is called $\text{Forb}(H)$.

$\chi(\text{Forb}(H)) = max_{G \in \text{Forb}(H)} \chi(G)$.

Question : For which graph H $\chi(\text{Forb}(H))$ is bounded.

Large chromatic number and induced subgraphs

- Induced subgraphs of cliques are cliques and
- there exists graphs with large chromatic number and girth.

So for any graphs H, $\chi(\text{Forb}(H)) = +\infty$.

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- there exists graphs with large chromatic number and girth.

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What if no big cliques ?

Conjecture (Gyárfás, 1975 ; Sumner, 1981) : For every integer k and tree T, $\chi(\text{Forb}(T, K_k))$ is bounded.

In other words, if $G \in \text{Forb}(T)$, then its chromatic number cannot be too big compared to its chromatic number.

The Gyárfas-Sumner Conjecture

A class of graphs $\mathcal{C}$ is *χ-bounded* if there exists a function f such that for all $G \in \mathcal{C}$, $\chi(G) \leq f(\omega(G))$.

Conjecture (Gyárfás, 1975 ; Sumner, 1981) : for every tree T, Forb(T) is χ-bounded.

Some partial results :

- Forb(P_k) is χ-bounded (Gyárfas, 1980),
- Forb(S_k) is χ-bounded (Gyárfas, 1980),
- Forb(T) is χ-bounded, for trees of radius 2 (Kierstead and Penrice, 1994)
- Forb*(T) is χ-bounded (Scott).

Outline

1. Subgraphs of graphs with large chromatic number
2. Subdigraphs of digraphs with large chromatic number
3. Induced subgraphs of graphs with large chromatic number
4. Induced subdigraphs of digraphs with large chromatic number
   - Forbidding oriented paths
   - Forbidding oriented stars

The oriented case

Question : for which oriented tree T is Forb(T) χ-bounded ?
i.e. for all $D \in$ Forb(T), there is a function f such that $\chi(D) \leq f(\omega(D))$.

Tournament = orientation of a complete graph.

$tt(D)$ = order of a largest transitive tournament in D.

Theorem (Erdős, Moser, 1964) : for any tournament T, $tt(T) \geq log(|V(T)|) + 1$.

So Forb(T) is χ-bounded is equivalent to :

- there is a function g such that $\chi(D) \leq g(tt(D))$ for all $D \in$ Forb(T).
- for all integers k, χ(Forb(T, TT_k)) is bounded.

Forbidding oriented paths

Theorem (Gyárfas, 1987) : for any integer k, $\mathrm{Forb}(P_k, K_k)$ is bounded.

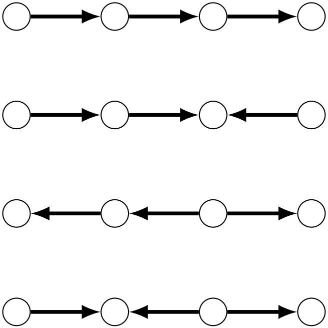
Forbidding oriented paths

Theorem (Gyárfas, 1987) : for any integer k, Forb(P_k, K_k) is bounded.

What about oriented paths ?

Forbidding oriented paths

Four ways to orient $\mathbf{P_3}$:



$P^+(3)$

$P^+(2,1)$

$P^-(2,1)$

$P^+(1,1,1)$

Observe that

$$\mathrm{Forb}(P_3) = \mathrm{Forb}(P^+(3), P^+(2,1), P^-(2,1), P^+(1,1,1))$$

Forbidding two induced oriented paths of length 3

Theorem :

(a) Forb($P^+(3), P^+(2,1)$) is χ-bounded.
(b) Forb($P^+(3), P^-(2,1)$) is χ-bounded.
(c) Forb($P^+(3), P^+(1,1,1)$) is χ-bounded.

Proof : (a), (b), (c) : No odd hole.

Theorem (Scott and Seymour, 2014) :
Odd-hole-free graphs are χ-bounded.

What about Forb($P^+(1,1,1), P^+(2,1), P^-(2,1)$) ?

Forbidding an induced oriented path of length 3

Theorem :

- $\chi(\text{Forb}(TT_3, \vec{C}_3, P^+(3))) = +\infty$.
- $\chi(\text{Forb}(TT_3, \vec{C}_3, P^+(1,1,1))) = +\infty$.

- $\chi(\text{Forb}(TT_3, \vec{C}_3, P^+(2,1)) = 3$.
- $\chi(\text{Forb}(TT_3, P^+(2,1))) = 4$.

If $\vec{P_k}$ contains $P^+(3)$ or $P^+(1,1,1)$, then $\text{Forb}(\vec{P}_k)$ is not χ-bounded.

Question : Are $\text{Forb}(P^+(2,2))$ and $\text{Forb}(P^+(1,2,1))$ χ-bounded ?

$\chi(\mathrm{Forb}(\overrightarrow{C}_{k\geq 3}, TT_3, C(3,1), C(2,2), P^+(1,1,1))) = +\infty$

Line-digraph of D, $L(D)$:
$V(L(D)) = A(D)$ and $A(L(D)) = \{(uv, vw))|uv, vw \in A(D)\}$.

Theorem : for any graph D,
$L(D) \in forb(TT_3, C(3,1), C(2,2), P^+(1,1,1))$.

Theorem : (Poljak and Rödl, 1981) $\chi(L(D)) \geq \log(\chi(D))$

$L(TT_n) \in forb(\overrightarrow{C}_{k\geq 3}, TT_3, C(3,1), C(2,2), P^+(1,1,1))$ and
$\chi(L(TT_n)) \geq \log(n)$.

$\chi(\text{Forb}(\{\overrightarrow{C}_{k\geq 3}, TT_3, P^+(3)\})) = +\infty.$

By induction on k, we construct $D_k \in \text{Forb}(\{\overrightarrow{C}_3, TT_3, P^+(3)\})$ such that $\chi(D_k) = k$.

$$\begin{aligned} S(v) &= \{u : \text{there is } (v,u)\text{-dipath of even length}\} \\ &= \{u : \text{there is a (v,u)-path of length 2}\} \end{aligned}$$

D_{k+1} obtained from D_k as follows :

- Take k copies $D_k^1, \ldots D_k^k$ of D_k.
- For every $(v_1, \ldots, v_k)$ with $v_i \in V(D_k^i)$:
  - add a new vertex $x = x(v_1, \ldots, v_k)$;
  - for all $1 \leq i \leq k$, add all arcs from $S(v_i)$ to x.

Forbidding oriented stars

Theorem (Gyárfas) : for all k, $\text{Forb}(S_t, K_k) \leq R(t, k)$.

Proof. If $D \in \text{Forb}(S_k)$, then all vertices have degree less than $R(k, \omega(D))$.

The Ramsey number $R(t, k)$ is the minimum number such that all graphs on $R(t, k)$ vertices either have a stable set of size t or a clique of size k.

Theorem : Forb($S_{0,k}$) is χ-bounded.

Proof. Ramsey.

Theorem : $\chi(\text{Forb}(TT_3, \overrightarrow{C}_3, S_{i,j})) \leq 2i + 2j - 2$.

Theorem : $\chi(\text{Forb}(TT_3, S_{i,j}))$ is bounded.

Conjecture : for all integers k, $\chi(\text{Forb}(S_{i,j}, TT_k)$ is bounded.

THANK YOU FOR YOUR ATTENTION