Inhomogeneous hypergraphs

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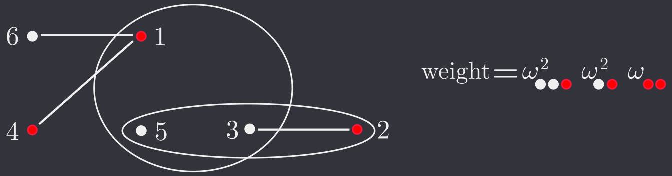
Introduction

Tool: Analytic combinatorics

Model: Inhomogeneous hypergraphs

Application: Constraint Satisfaction Problems (CSP)



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$$\exists x_1, \ldots, x_n,\ C_1\left(x_{r(1,1)}, \ldots, x_{r(1,a_1)}\right) \wedge \cdots \wedge C_m\left(x_{r(m,1)}, \ldots, x_{r(m,a_m)}\right)$$

Enumeration of satisfied instances.

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Generating function $$A(z) = \sum_{a \in \mathcal{A}} \frac{z^{|a|}}{|a|!},$$

combinatorial description $\rightarrow$ equation on the generating function,

asymptotics and limit laws $\leftarrow$ analytic properties.

Inhomogeneous graphs

"The evolution of graphs may be considered as a rather simplified model of the evolution of certain communication nets [...]. Of course, if one aims at describing such a real situation, one should replace the hypothesis of equiprobability of all connections by some more realistic hypothesis."

– Erdős, Rényi (1959)

Probabilistic model introduced by Söderberg, extended by Bollobás, Janson, Riordan. Enumerative model analyzed by E.d.P. and Ravelomanana.

- The model inputs a parameter $R \in \mathrm{Sym}_q(\mathbb{R}_{\geq 0})$,
- each vertex v receives a color $t(v)$ in $\{1, \dots, q\}$,
- each edge (v, w) has a weight $R_{t(v),t(w)}$.

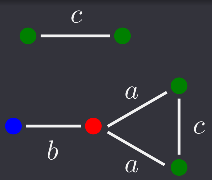
Inhomogeneous graphs are counted with a weight

$$\mathrm{weight}(G) = \prod_{(v,w) \in \mathrm{edges}(G)} R_{t(v),t(w)}.$$

Inhomogeneous graphs

$$R = \begin{pmatrix} 0 & a & b \\ a & c & 0 \\ b & 0 & 0 \end{pmatrix}$$

$G =$



$$\text{weight}(G) = a^2bc^2$$

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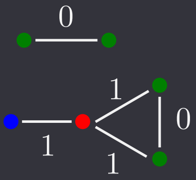
Example: properly q-colored graphs

A graph is properly q-colored if each vertex has a color in $\{1, \ldots, q\}$, and no edge links two vertices having the same color.

Bijection between inh. graphs and properly q-colored graphs

$$R = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$G =$



$\mathrm{weight}(G) = 0$

Asymptotics for n vertices and m edges

$$\frac{n^{2m}}{2^m m!} e^{\left(\frac{m}{n}\right)^2 \frac{q}{q-1}} \left(1 + \frac{2}{q-1}\frac{m}{n}\right)^{-\frac{q-1}{2}} \left(1 - \frac{1}{q}\right)^m q^n \left(1 + o(1)\right)$$

when m/n has a positive limit. A result already obtained by Wright (1972).

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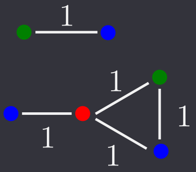

$$\text{weight}(G) = 1$$

Asymptotics for n vertices and m edges

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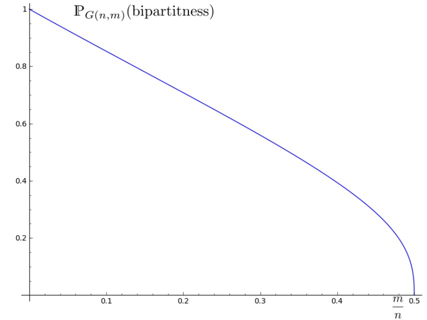
2-colorable graphs

A 2-colorable graph with c components has 2^c proper colorations.

Structure of inh. graph $\rightarrow$ enumeration of 2-colorable graphs.

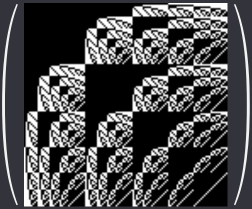


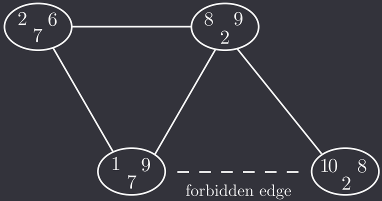
result already obtained by Pittel and Yeum (2004).

Example: friendship graphs

Each vertex has r hobbies among a set of size s, two vertices can be linked only if they share at least t hobby.

$R =$





$$(r, s, t) = (3, 10, 1).$$

R is the adjacency matrix of the complement of the Kneser graph.

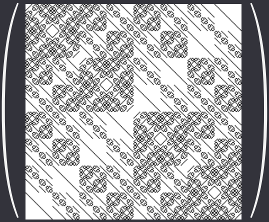
Example: systems of 2-equations

Consider a finite set E of triplets from $\mathbb{F}_d$, and a system

$$\exists x_1, \ldots, x_n \in \mathbb{F}_d, \quad \begin{cases} a_1 x_{e(1,1)} + b_1 x_{e(1,2)} & = & c_1, \\ & \vdots & \\ a_m x_{e(m,1)} + b_m x_{e(m,2)} & = & c_m. \end{cases}$$

The probability of satisfiability is expressed using the enumeration of inhomogeneous graphs

$R =$



For 2QXorSAT, R is a Hamming-like matrix.

Constraint Satisfaction Problems

A CSP is a set of Boolean functions taking value in a finite set.

An instance is a formula

$$\exists x_1, \dots, x_n,\ C_1\left(x_{r(1,1)}, \dots, x_{r(1,a_1)}\right) \wedge \cdots \wedge C_m\left(x_{r(m,1)}, \dots, x_{r(m,a_m)}\right)$$

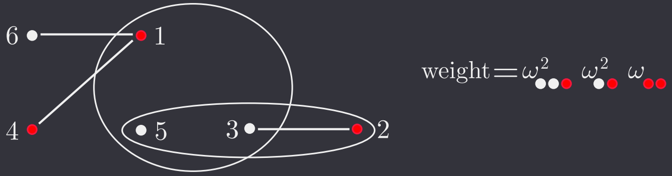
A satisfied instance is a pair instance-solution.

Bijection between satisfied instances of CSP with clauses of arity 2, and inhomogeneous graphs

variable	vertex,
constraint	edge,
values	colors,
number of clauses satisfied by (i, j) or (j, i)	$R_{i,j}$.

Inhomogeneous hypergraphs

- Each vertex v receives a color $t(v)$ in $\{1, \ldots, q\}$,
- edges can contain more than 2 vertices,
- the weight of an edge depends of the colors it contains.



$$\text{orderings}(G) = |\{((5,2,3),(4,1),(2,3),(1,5,3),(1,6)),\ldots\}|,$$

$$\text{weight}(G) = \frac{\text{orderings}(G)}{|E(G)|!\prod_{e \in E(G)} |e|!} \prod_{e \in E(G)} \omega_{\bar{t}(e)}$$

Analytic combinatorics of inhomogeneous hypergraphs

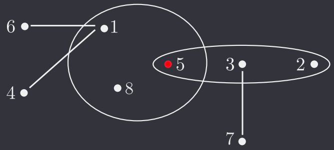
Many parameters: for all $\bar{t} \in \mathbb{N}^q$, weight $\omega_{\bar{t}}$ for the edges that contain t_i vertices of type i.

$$\Omega(x_1, \dots, x_q) = \sum_{t_1, \dots, t_q \geq 0} \omega_{t_1, \dots, t_q} \frac{x_1^{t_1}}{t_1!} \cdots \frac{x_q^{t_q}}{t_q!}.$$

Miracle: statistics of the model reduced to analytic properties of Ω.

Example: tree with root of type i = root and set of edges with 1 vertex of type i removed and the other replaced by rooted trees.

$$T_i(z) = \sum_n \left(\begin{array}{l} \text{nb trees of size } n \\ \text{root of color } i \end{array} \right) \frac{z^n}{n!}.$$



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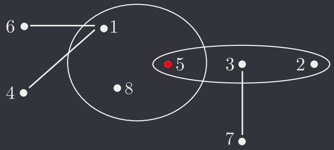

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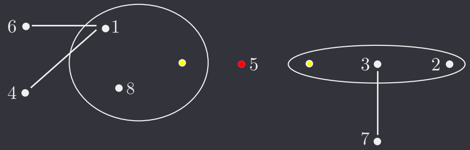

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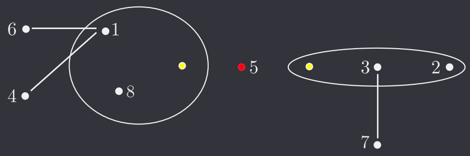
$$T_i(z) = z e^{\partial_i \Omega(T_1(z), \dots, T_q(z))}.$$



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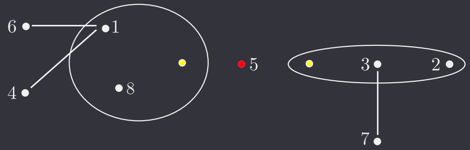


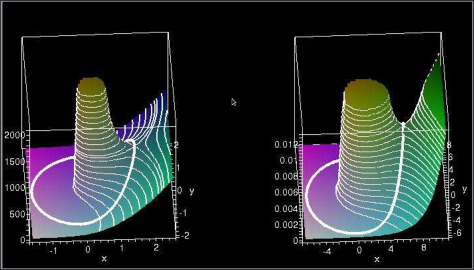
Trees and unicycles

Unrooted trees: $U(z) = \sum_i T_i + \Omega(\bar{T}) - \bar{T}\bar{\partial}\Omega(\bar{T})$

Unicycles: $V(z) = -\frac{1}{2}\log\left(\det\left(\mathrm{Id} - \mathrm{diag}(\bar{T})\mathcal{H}_\Omega(\bar{T})\right)\right)$

Trees and unicycles: excess $k = \sum_{e \in \mathrm{edges}}(|e| - 1) - n$

$$n![z^n]\frac{U(z)^{-k}}{(-k)!}e^{V(z)} = \frac{n!}{2i\pi}\oint \frac{U(z)^{-k}}{(-k)!}e^{V(z)}\frac{dz}{z^{n+1}}$$



(plot from Flajolet, Sedgewick 2009)

Asymptotics of all inhomogeneous hypergraphs

Sum of the weights of inhygraphs with n vertices and excess k

$$[y^{k+n}] \sum_{n_1+\dots+n_q=n} \binom{n}{\bar{n}} \prod_{\bar{t}\in\mathbb{N}^q} \left(1+\omega_{\bar{t}}\, y^{t_1+\dots+t_q-1}\right)^{\binom{n_1}{t_1}\cdots\binom{n_q}{t_q}}$$

Approximations: coefficient extraction, Stirling, integral for the sum

$$\sim C_{n,k} \int_{\substack{x_1+\dots+x_q=1\\ \bar{x}\in[0,1]^q}} A(\bar{x}) e^{n\Phi(\bar{x})} d\bar{x}$$

where $\Psi(\bar{x}) = \frac{\bar{x}\bar{\partial}\Omega(\bar{x})-\Omega(\bar{x})}{\sum_{i=1}^{q} x_i}$, $\Psi(\zeta\bar{x}) = 1+\frac{k}{n}$

and $\Phi(\bar{x}) = \bar{x}\left(\log(\bar{\nu}) - \log(\bar{x})\right) + \frac{\Omega(\zeta\bar{x})}{\zeta} - \left(1+\frac{k}{n}\right)\log(\zeta)$.

trees and unicycles — bounded excess components — giant component

0 — $\Lambda + O(n^{-1/3})$ — $\Gamma_1 \cdots \Gamma_q$ — $\frac{\sum_{e\in \text{edges}}(|e|-1))}{n}$

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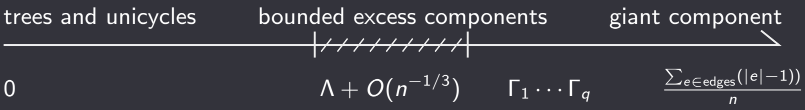

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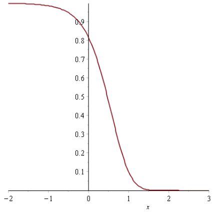

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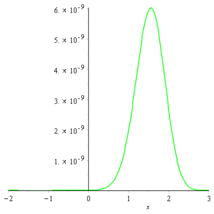

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Forthcoming research

Applications will provide new inhygraphs properties to investigate

- structure of graphs with a giant component
  generating function of connected graphs with the same density of edges,
- graphs with forbidden subgraphs
  satisfiability threshold of 2-SAT,
- data base modelling using inhomogeneous hypergraphs
  analysis of data mining algorithms (Peps HYDrATA).