# Community detection and Role extraction in Networks


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EPL - ICTEAM


Complex Networks
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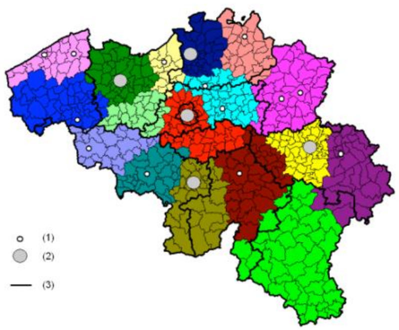
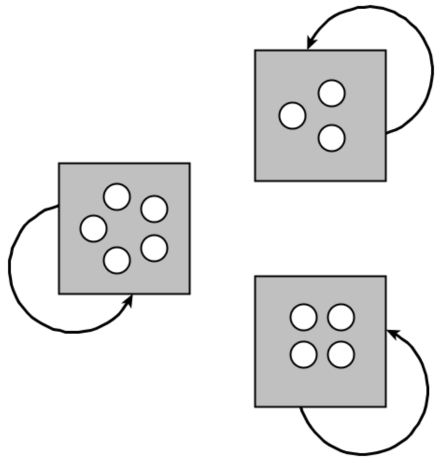
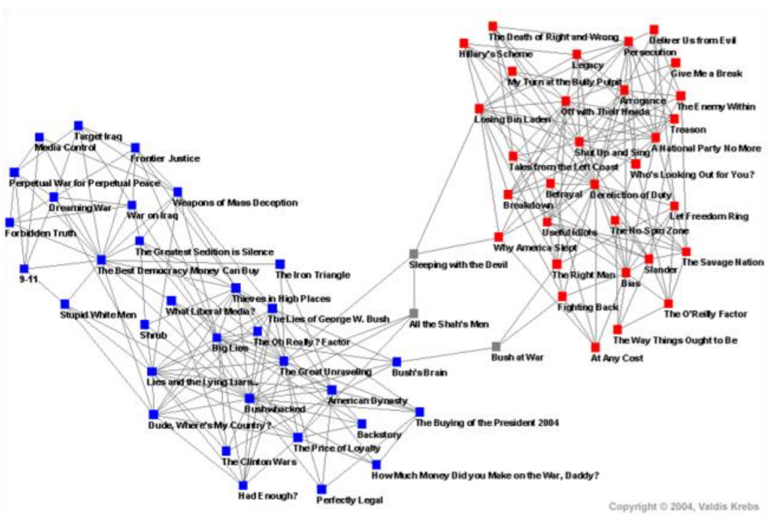
# Networks Topology

## Community Structures

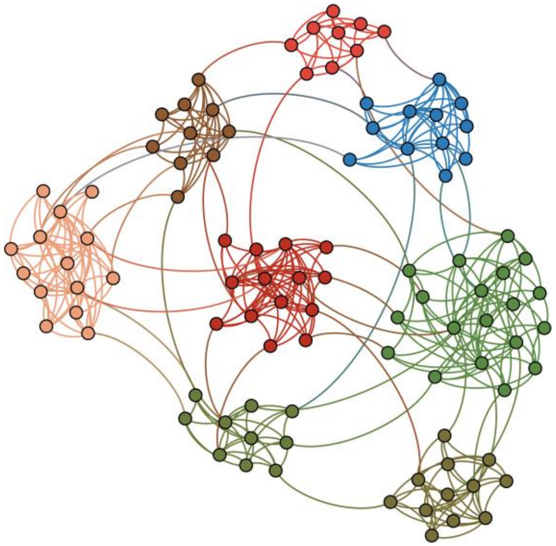





# Networks Topology

## Community Structures

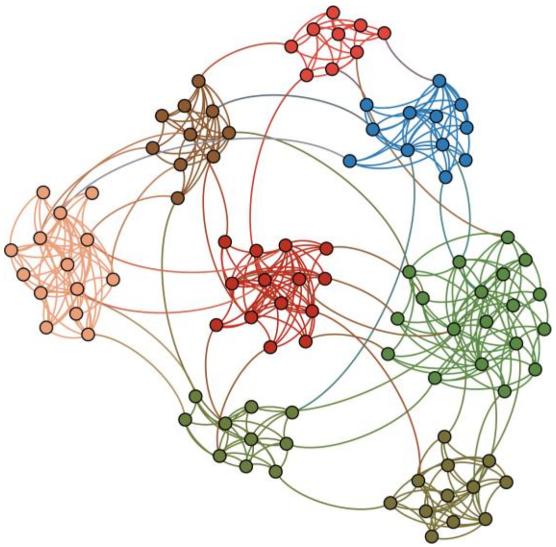


$$H(\sigma) = -H_0 - \sum_{i,j \in V} \left[\alpha_{ij} A(i,j) - \beta_{ij}\right] \delta\left(\sigma_i, \sigma_j\right)$$

*Reichardt & Bornholdt (2006)*

| | Unweighted network | Weighted network |
|---|---|---|
| Reichardt & Bornholdt | $\alpha_{ij} = 1$ | $\alpha_{ij} = w(i,j)$ |
| | $\beta_{ij} = \gamma_{RB} p_{ij}$ | $p_{ij} = \frac{mn_c^2}{n^2}$ |
| Newman & Girvan (modularity) | $\alpha_{ij} = 1$ <br> $\beta_{ij} = \frac{k_i^{out} k_j^{in}}{m}$ | $\alpha_{ij} = w(i,j)$ <br> $\beta_{ij} = \frac{s_i^{out} s_j^{in}}{m_w}$ |
| Traag et al. (CPM) | $\alpha_{ij} = 1$ | $\alpha_{ij} = w(i,j)$ |
| | $\beta_{ij} = \gamma_{CPM}$ | |
| Ronhovde & Nussinov | $\alpha_{ij} = 1 + \gamma_{RN}$ | $\alpha_{ij} = w(i,j) + \gamma_{RN}$ |
| | $\beta_{ij} = \gamma_{RN}$ | |
| Raghavan et al. (label propagation) | $\alpha_{ij} = w(i,j)$ $\beta_{ij} = 0$ | |

# Networks Topology
## Community Structures



$$H_M(\sigma) = q_{out} H(q_{out}) + \sum_{k=1}^{m} q_{k,in} H(q_{k,in}).$$

*Rosvall & Bergstrom (2008)*

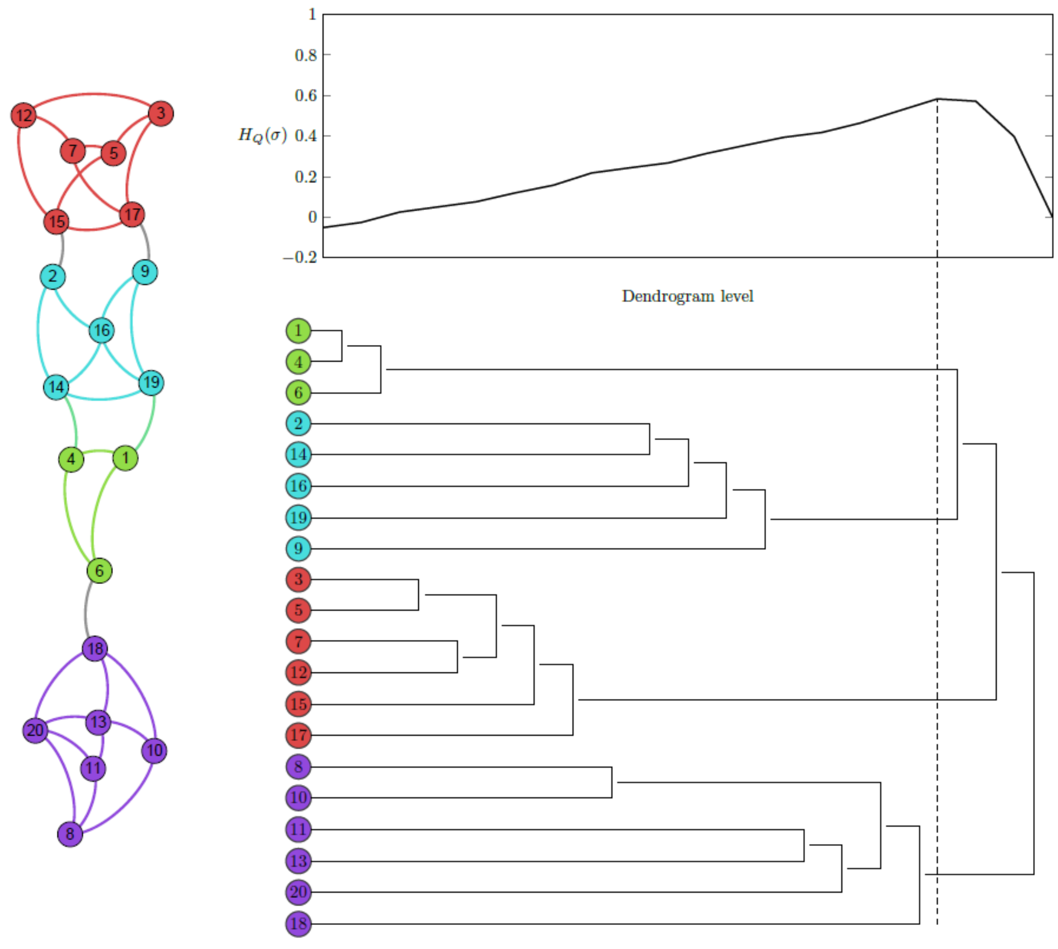
$$H_S(\sigma) = -\log \sum_{j=f}^{\min(F,m)} \frac{\binom{F}{j}\binom{M-F}{m-j}}{\binom{M}{m}}$$

*Aldecoa & Marin (2011)*

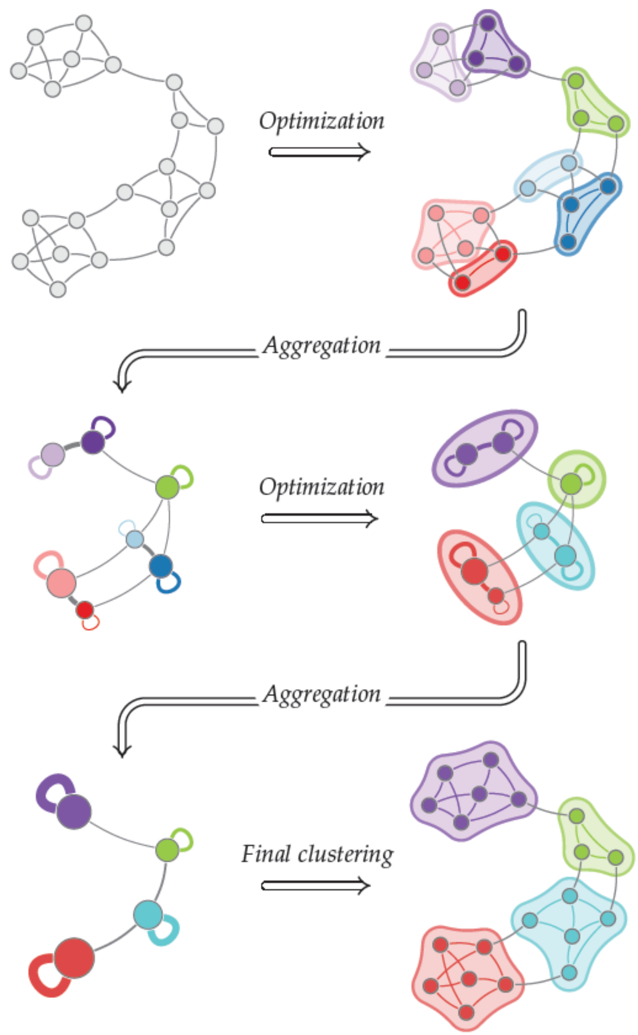
# Community Detection Algorithm

- Simulated annealing (SA)
  - *Kirkpatrick, Gelatt & Vecchi (1983)*
- Fast modularity
  - *Clauset, Newman & Moore (2004)*
- Fast modularity (+ TCER)
  - *Schuetz & Caflisch (2008)*
- Label propagation (LP)
  - *Raghavan, Albert & Kumara (2007)*
- Louvain Method (LM)
  - *Blondel, Guillaume, Lambiotte, et al. (2008)*
- Infomap
  - *Rosvall & Bergstrom (2010)*

# Community Detection Algorithm



# Community Detection Algorithm



# Fast community extraction

**Input** : a graph $G(V, E)$
**Output** : a community partition matrix $C \in \mathbb{R}^{k \times n}$

Initialize $C = I_n,\ C_t = 0,\ G_t = G$

**while** $C_t \neq I$ **do**
  $C_t \leftarrow \textsc{Assign}(G_t)$
  $C_t \leftarrow \textsc{Positive}(C_t, G_t)$
  **while** $\exists\, i \in V_t, c \in C_t$ with $\Delta H\,(c_i \rightarrow i \rightarrow c) > 0$ **do**
    $C_t \leftarrow \textsc{Maximal}(C_t, G_t)$
    $C_t \leftarrow \textsc{Positive}(C_t, G_t)$
  $G_t \leftarrow \textsc{Aggregate}(G_t, C_t)$
  $C = C_t C$

# Fast community extraction

---

**function** ASSIGN$(G(V, E))$
    **for all** $i \in V$ **do**
        $a(i) = \arg\max_j \Delta H(i \to j)$
    **end for**
    $T \leftarrow \text{graph}(V, \{(i, a(i))\ \forall i\})$
    $C_t \leftarrow WCC\ (T)$
    **return** $C_t$

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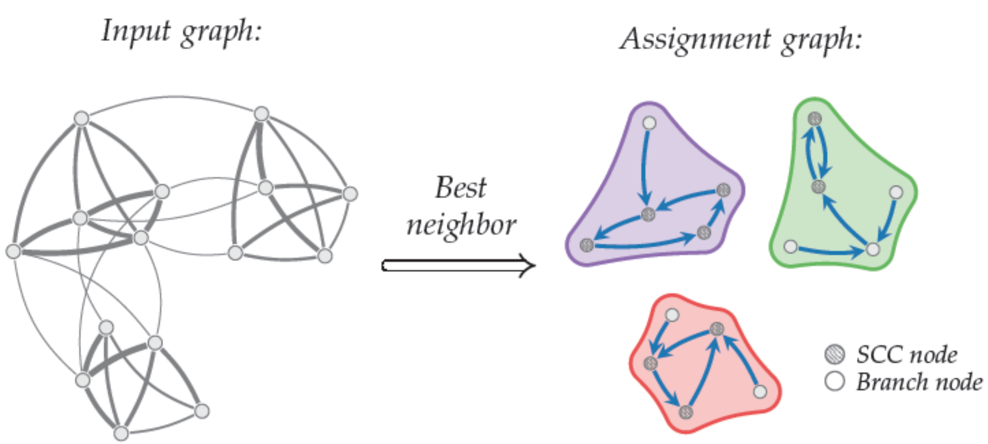
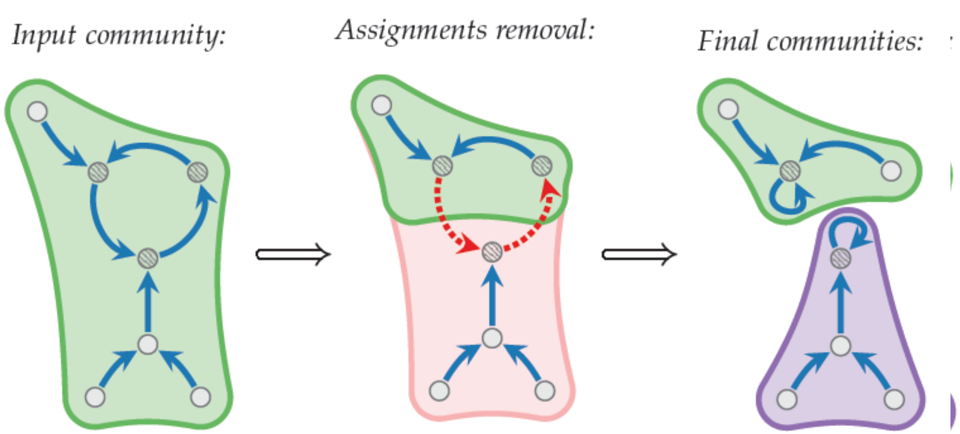


# Fast community extraction

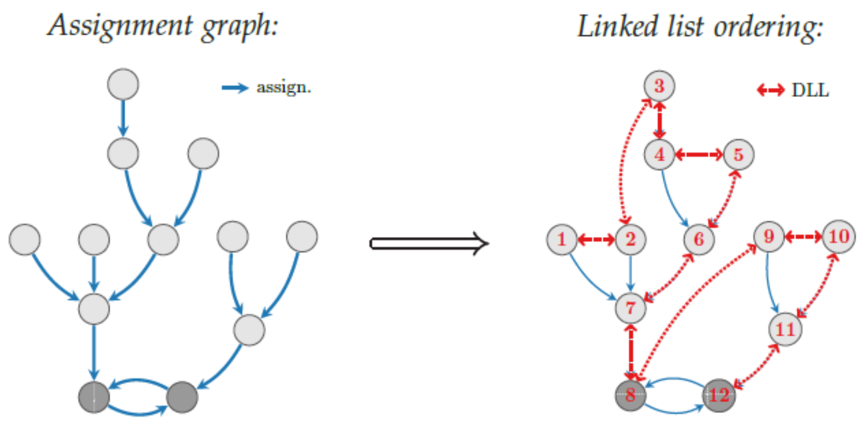
---

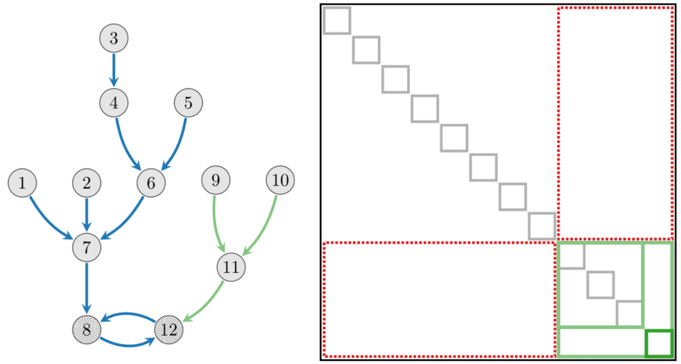
**function** $\textsc{Positive}(C_t, G(V,E))$
  **for all** $i \in V$ **do**
    $g(i) = -\Delta H\left(c_i \to i \to \{\}\right)$
  **while** $\exists i \in c_i$ with $g(i) < 0$ **do**
    $c_1, c_2 \leftarrow \textsc{Split}(c_i)$
    **for all** $j \in c_1 \cup c_2$ **do**
      $g(j) = -\Delta H\left(c_j \to j \to \{\}\right).$
    $C_t = C_t \setminus \{c_i\} \cup \{c_1, c_2\}$
  **return** $C_t$

---



# Fast community extraction



# Fast community extraction

---

**function** MAXIMAL($C_t, G(V,E)$)
  $C = C_t$
  **for all** $i \in V$ **do**
    $c_i^* = \arg\max_c \Delta H(c_i \to i \to c)$
  **for all** $i \in V$, if $c_i^* \neq c_i$ **do**
    draw $p(i)$ uniform $\in [0,1]$
    **if** $p(i) < p$ **then**
      $b(i) = branch(i)$
      **if** $\Delta H(c_i \to b(i) \to c_i^*) > 0$ **then**
        $a(i) = \arg\max_{j \in c_i^*} \Delta H(i \to j)$
        $C \leftarrow insert\left(b(i), c_i^*\right)$.
  **return** $C$

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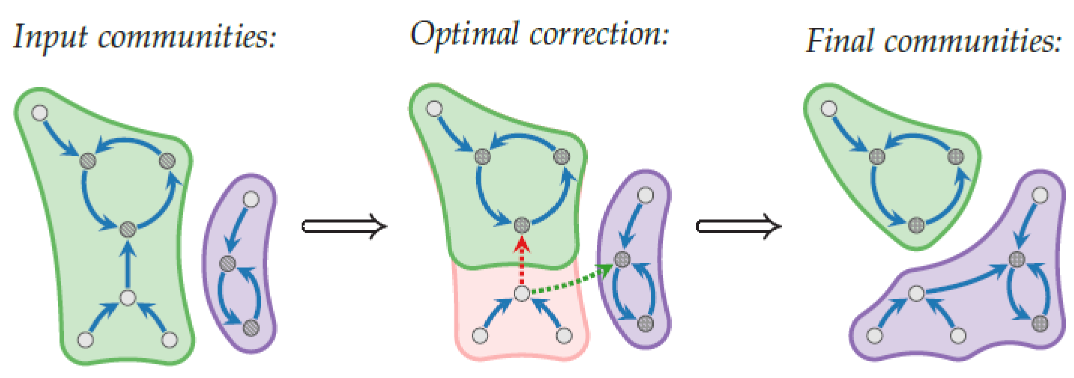

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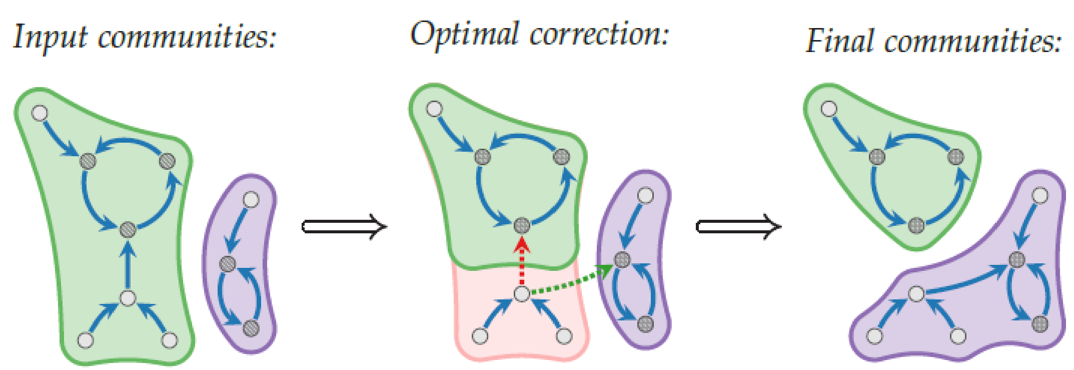
**return** $C$



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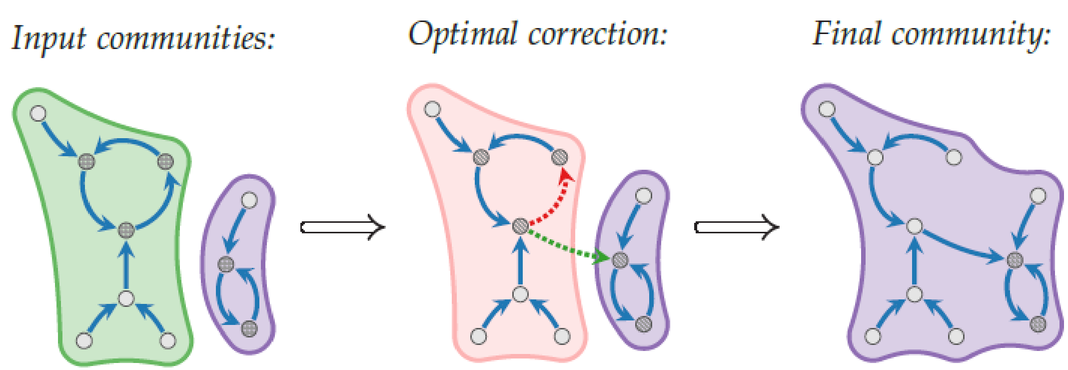

# Fast community extraction

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  **for all** $i \in V$ **do**
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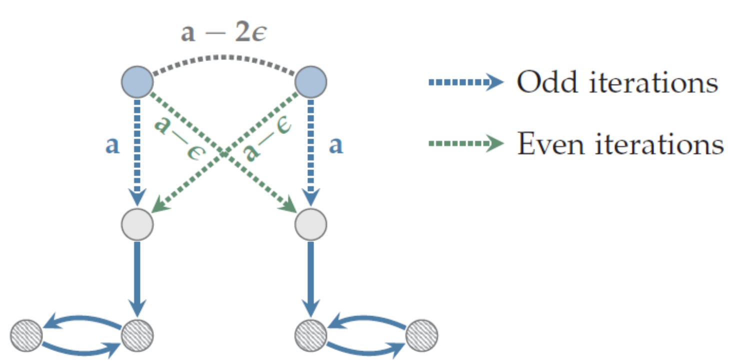
  **return** $C$

---



# Performance & Accuracy

## LFR benchmark model

*Lancichinetti, Fortunato & Radicchi (2008)*

$$k_i \backsim k^{-\tau_1} \qquad n_c \backsim n^{-\tau_2}$$

$$\langle k_{int} \rangle = (1 - \mu_T) \langle k \rangle \,,$$
$$\langle k_{ext} \rangle = \mu_T \langle k \rangle \,.$$

$$s_i^{int} = (1 - \mu_W)\, k_i^{\beta},$$
$$s_i^{ext} = \mu_W k_i^{\beta}.$$

$$\left\langle w^{int} \right\rangle = \frac{(1-\mu_W)\langle s \rangle}{(1-\mu_T)\langle k \rangle} = \frac{(1-\mu_W)}{(1-\mu_T)} \langle k \rangle^{\beta-1} \,,$$
$$\left\langle w^{ext} \right\rangle = \frac{\mu_W \langle s \rangle}{\mu_T \langle k \rangle} = \frac{\mu_W}{\mu_T} \langle k \rangle^{\beta-1} \,,$$

## Normalized mutual information

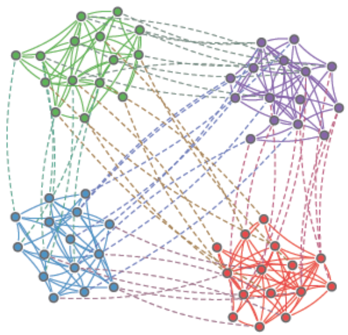
*Danon, Diaz-Guilera, Duch, et al. (2005)*

$$NMI(X,Y) = \frac{2\, I(X,Y)}{H(X) + H(Y)}.$$

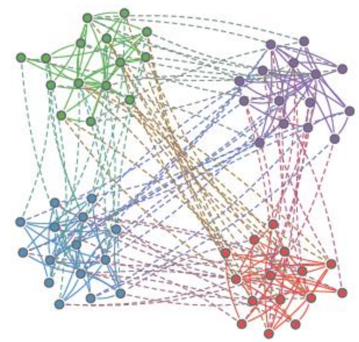
# Performance & Accuracy

## LFR benchmark model

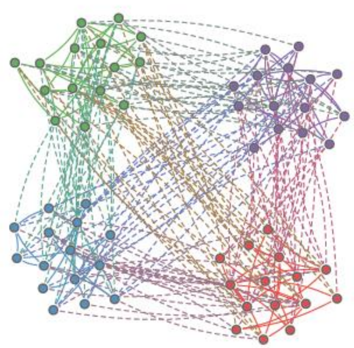
*Lancichinetti, Fortunato & Radicchi (2008)*



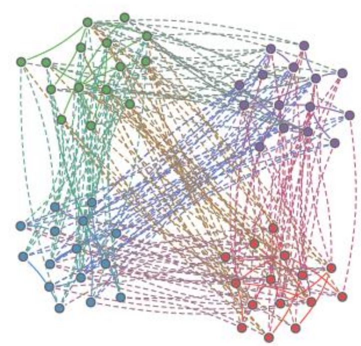
(a) $\mu_T = 0.2$



(b) $\mu_T = 0.4$

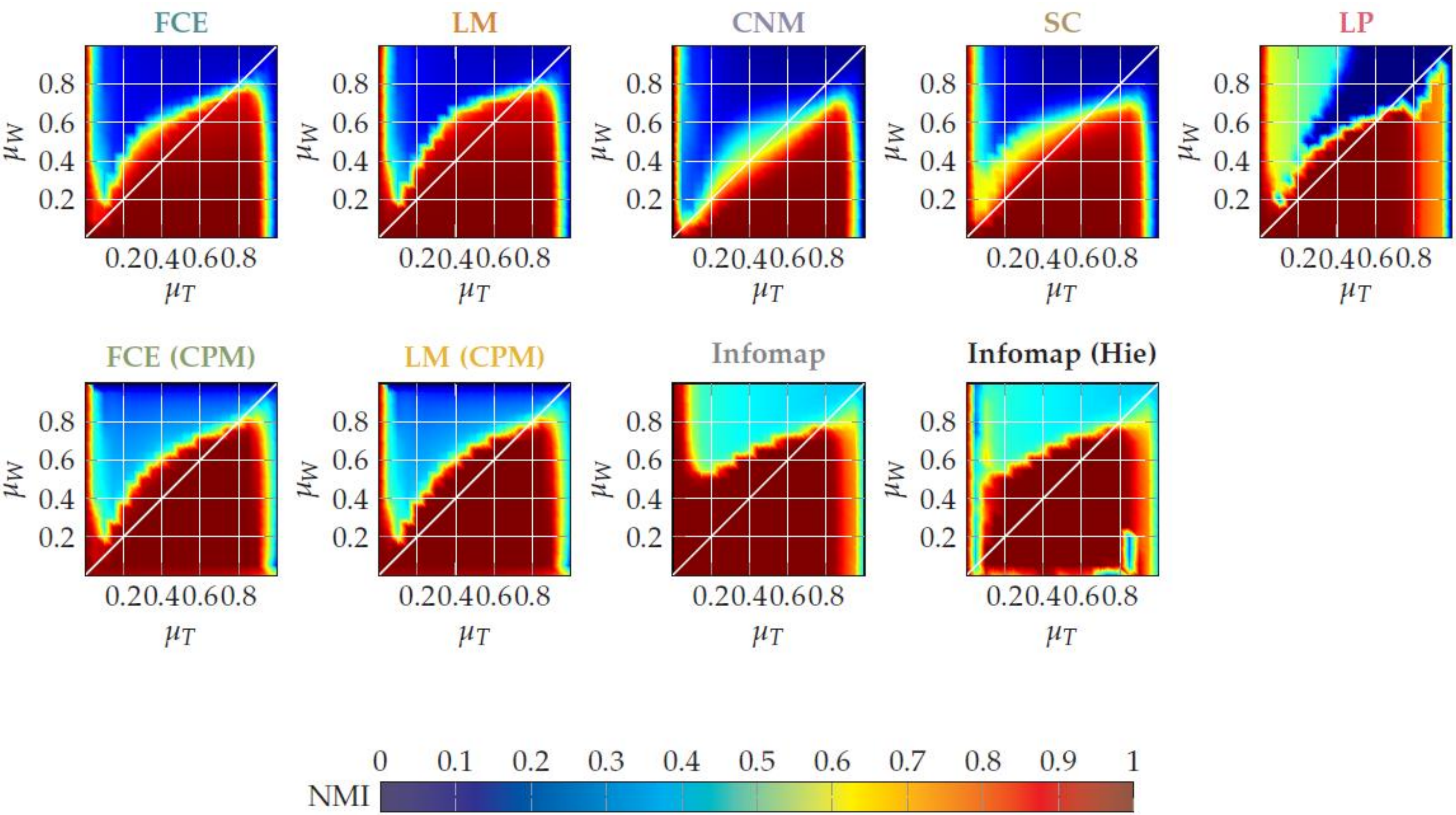


(c) $\mu_T = 0.6$

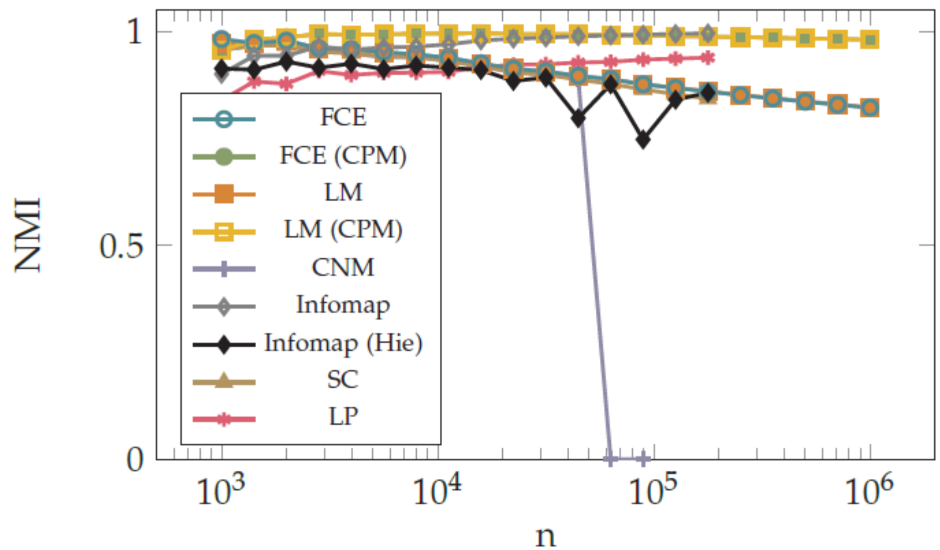


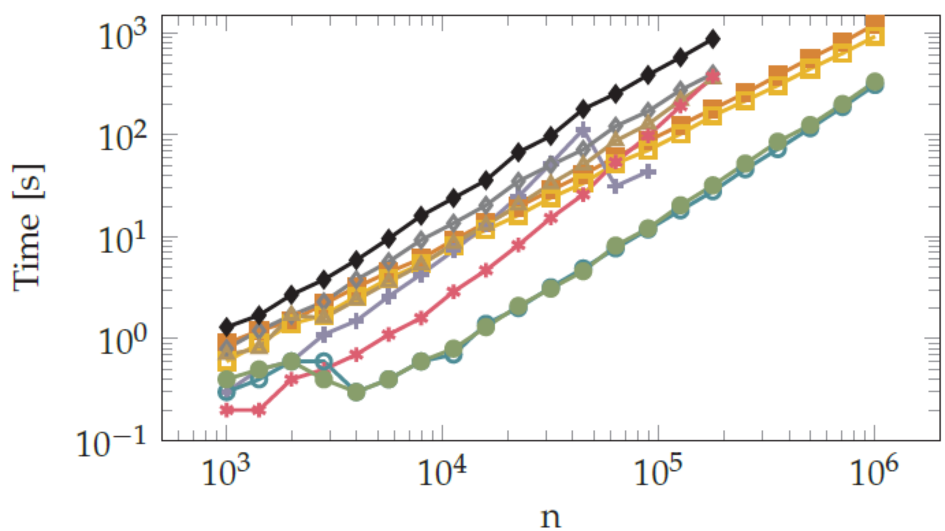
(d) $\mu_T = 0.8$

# Performance & Accuracy



# Performance & Accuracy



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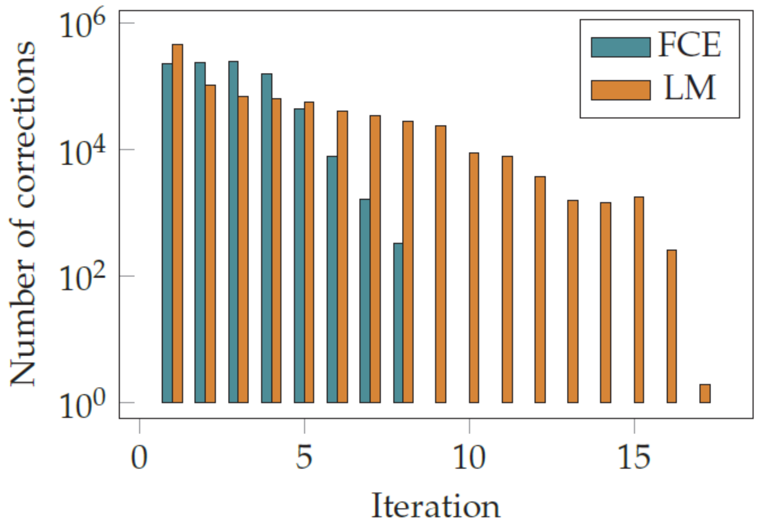
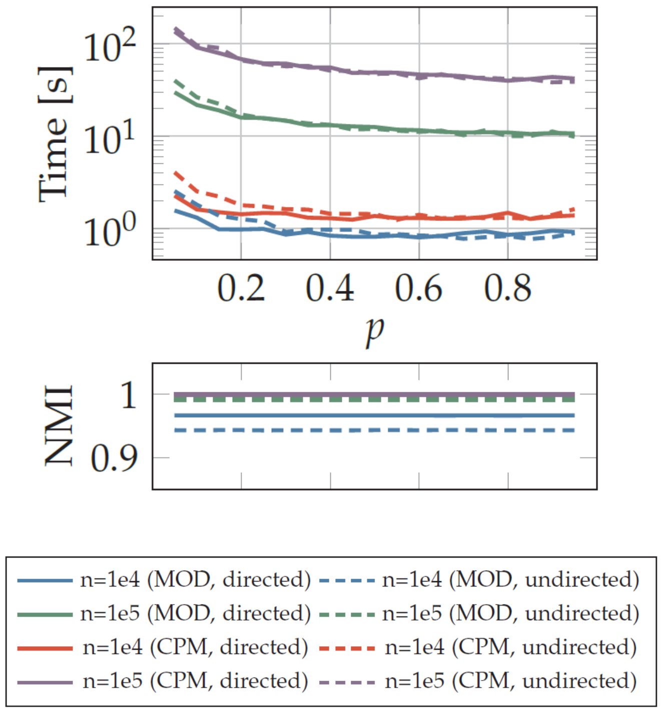


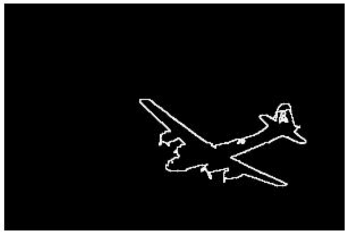



# Application to image processing

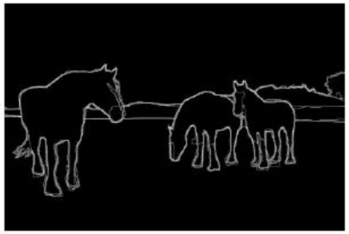
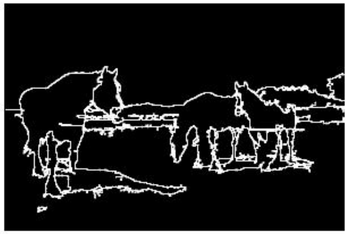
$$w(i,j) = \begin{cases} e^{\frac{d(i,j)^2}{\sigma_x^2}} e^{\frac{|F(i)-F(j)|^2}{\sigma_i^2}} & \text{if } d(i,j) < d_{max}, \\ 0 & \text{otherwise} \end{cases}$$

$$Q_\Lambda(\sigma) = \frac{1}{m} \sum_{i,j \in V} \left[ W - \frac{S \Lambda S}{m} \right]_{(i,j)} \delta(\sigma_i, \sigma_j)$$





Input picture Human benchmark FCE segmentation

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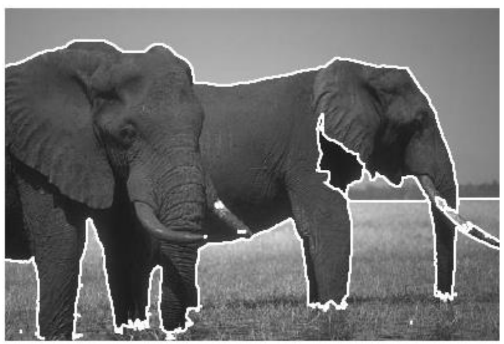

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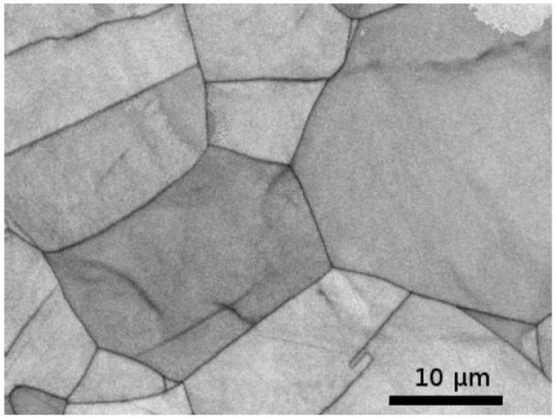
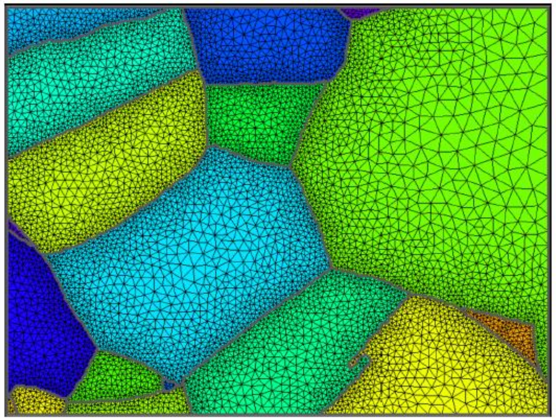
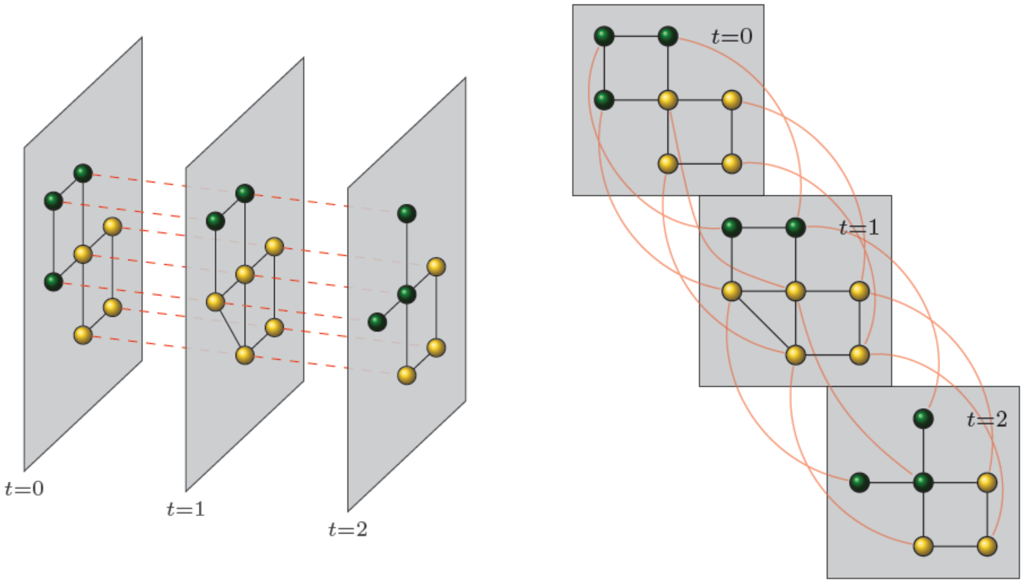


# Application to video tracking



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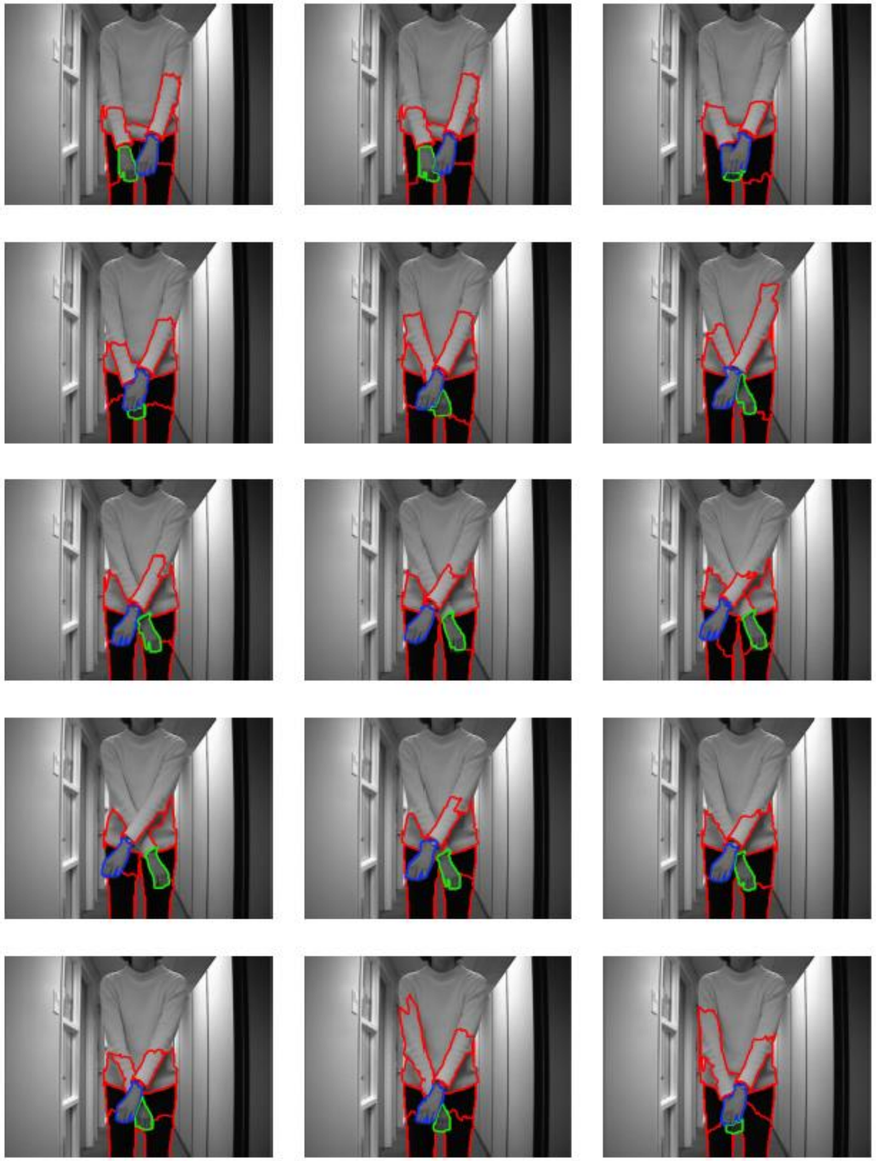
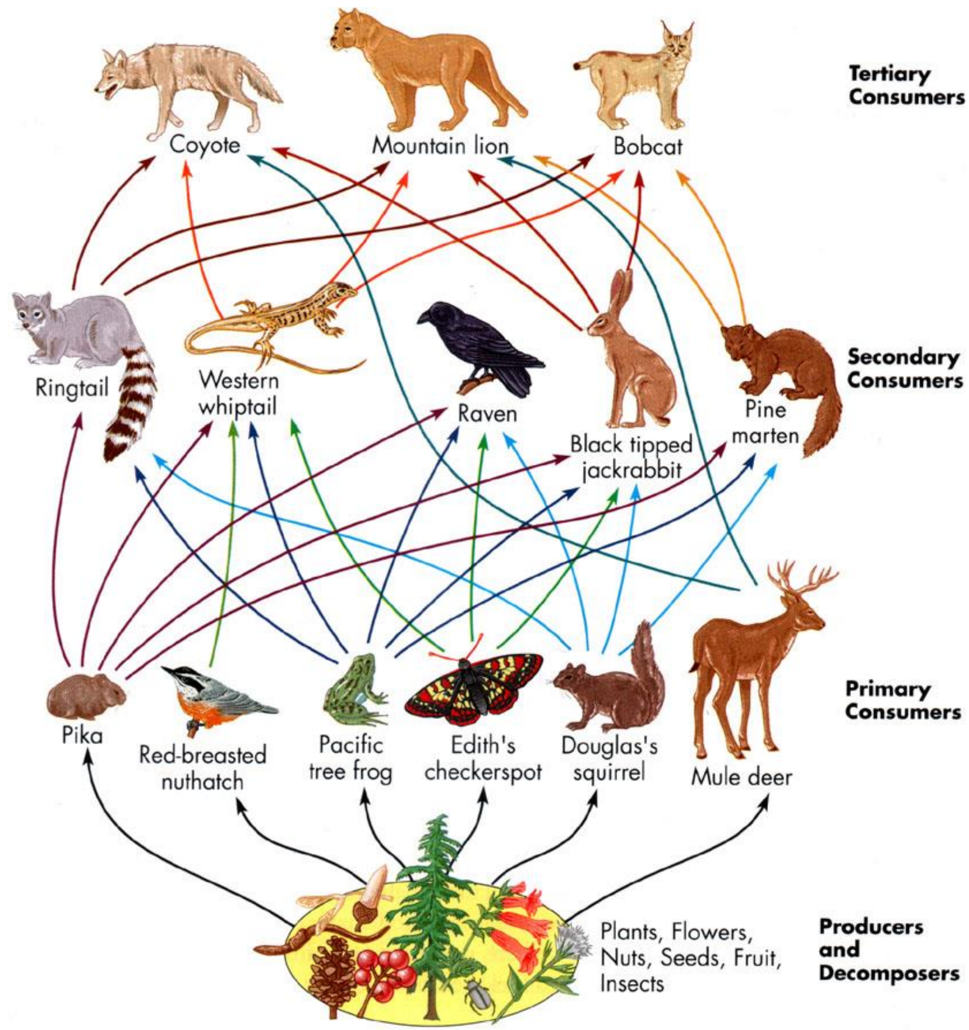
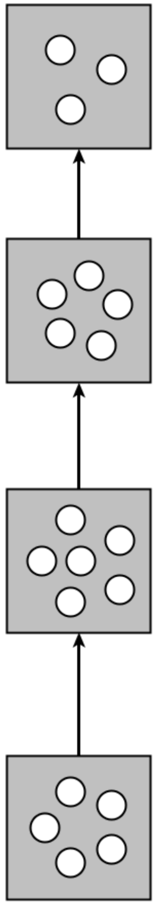


# Networks Topology

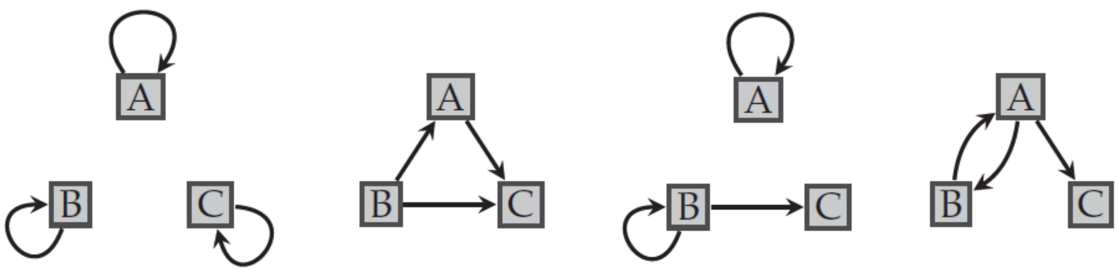
## Role Structure

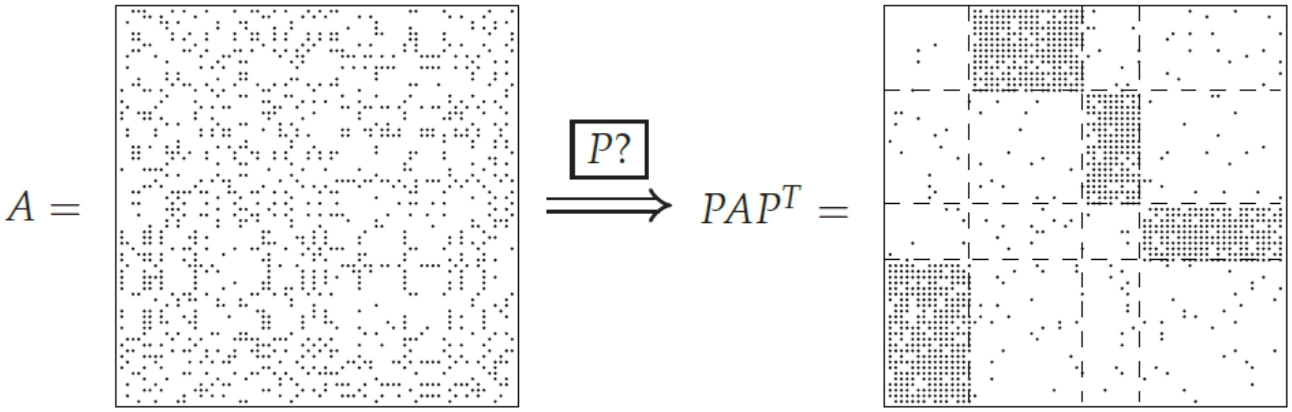


# Networks Topology

## Role Structure



# Role modeling
## Pairwise node similarity

*Bondel, Gajardo, Heymans, et al. (2004)*

$$S_{k+1} = \frac{A\,S_k\,A^T + A^T\,S_k\,A}{\|A\,S_k\,A^T + A^T\,S_k\,A\|_F}.$$

*Cooper & Barahona (2011)*

$$X = \left[\beta A \mathbf{1} \mid \ldots \mid (\beta A)^{l_{max}} \mathbf{1} \mid \beta A^T \mathbf{1} \mid \ldots \mid \left(\beta A^T\right)^{l_{max}} \mathbf{1}\right]$$

$$S_A^{CB}(i,j) = \frac{x_i x_j^T}{\|x_i\|\,\|x_j\|}.$$

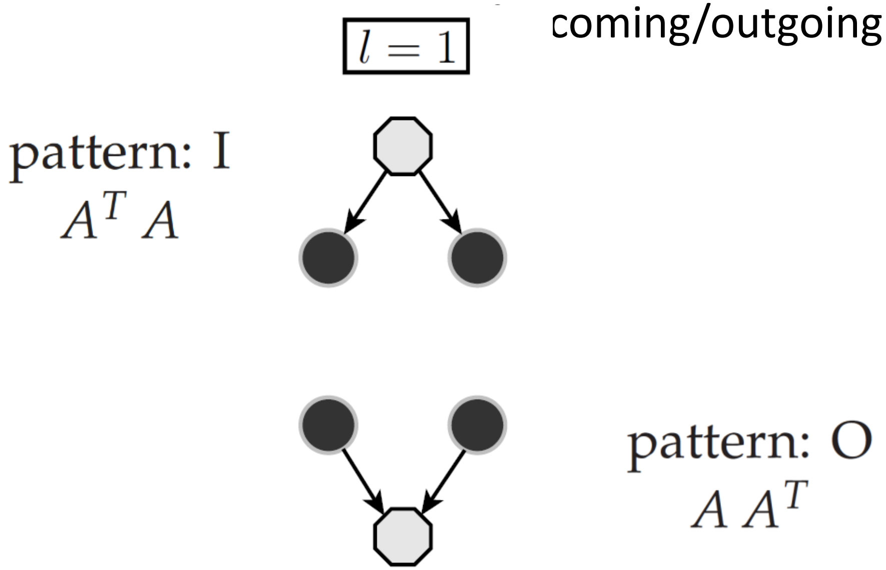
*Leicht, Holme & Newman (2006)*

$$S_A^L(i,j) = \delta(i,j) + \frac{m\lambda}{k_i^{out} k_j^{in}} \sum_{l=1}^{\infty} \left(\frac{\alpha}{\lambda}\right)^l \left[A^l\right](i,j)$$

# Role modeling
## Pairwise node similarity

✓ Groups of nodes sharing similar neighborhood patterns in the graph

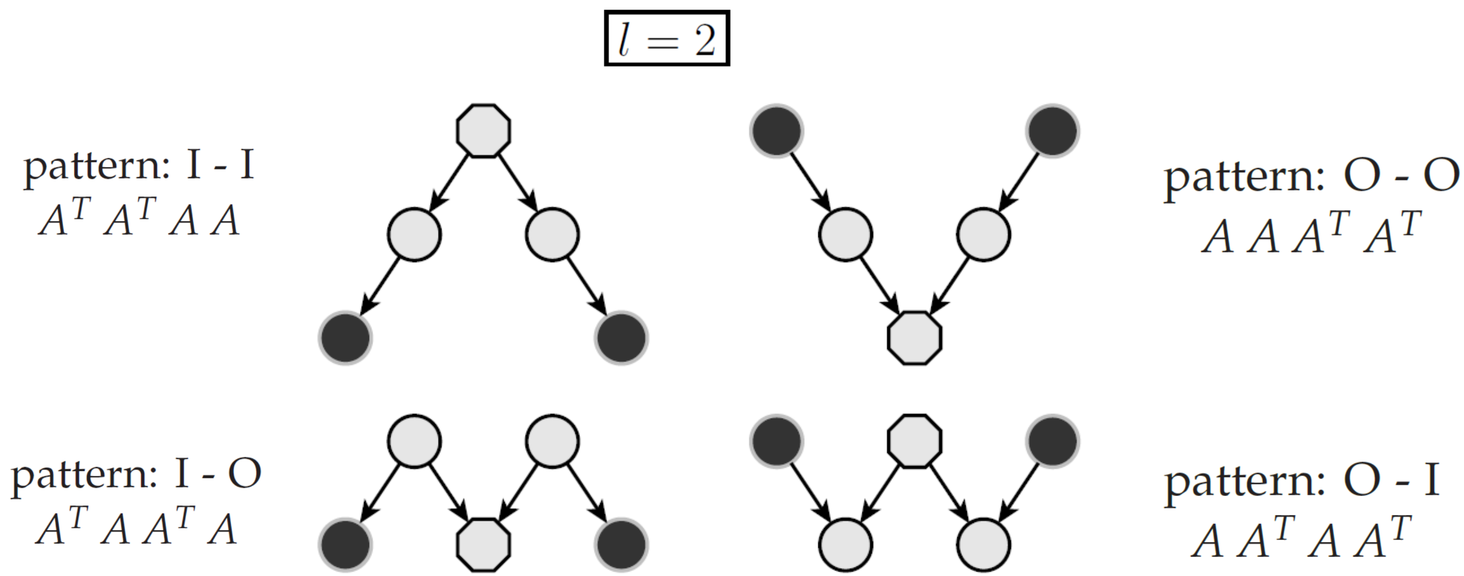
coming/outgoing edges




$$T_1 = AA^T + A^T A$$

# Role or Block modeling

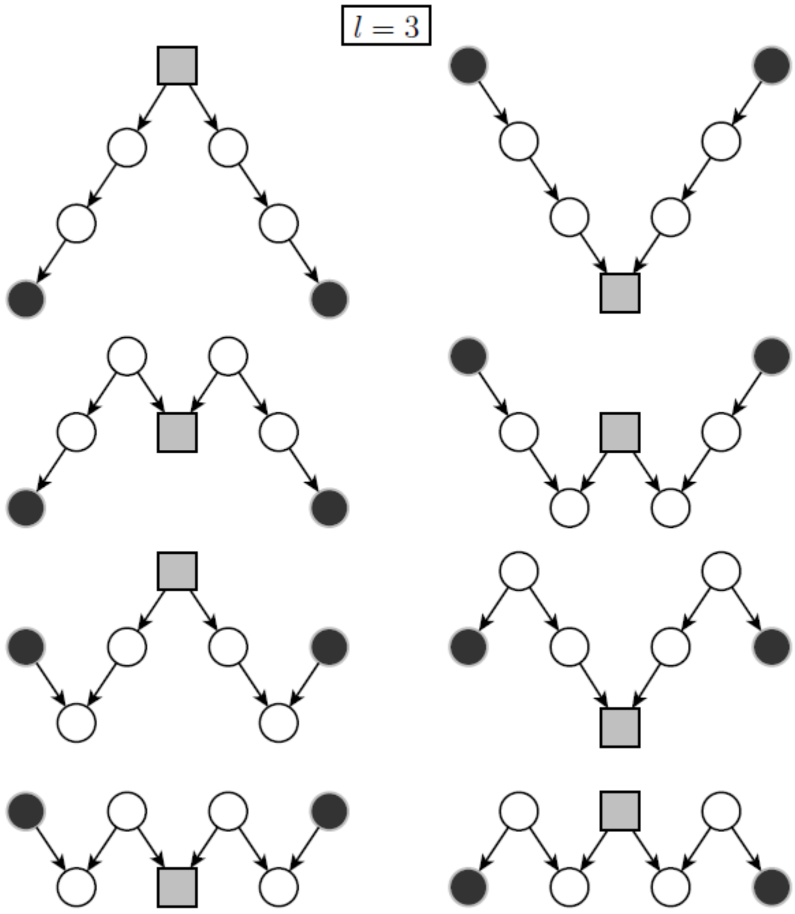
✓ Groups of nodes sharing similar neighborhood patterns in the graph




$$T_2 = AAA^TA^T + AA^TAA^T + A^TAA^TA + A^TA^TAA.$$

# Role or Block modeling

- ✓ Groups of nodes sharing similar neighborhood patterns in the graph



# Role or Block modeling
# Pairwise Similarity Measure

- ✓ Groups of nodes sharing similar neighborhood patterns in the graph

$$S = \sum_{\ell=1}^{\infty} \beta^{2(\ell-1)} T_{\ell}.$$

$$\Gamma_A : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n} : \Gamma_A[X] = AXA^T + A^T XA$$

$$T_1 = \Gamma_A[I],$$
$$T_2 = \Gamma_A[T_1] = \Gamma_A^2[I],$$
$$T_3 = \Gamma_A[T_2] = \Gamma_A^3[I],$$

$$S = \sum_{\ell=1}^{\infty} \beta^{2(\ell-1)} \Gamma_A^{\ell}[I],$$

# Role or Block modeling
# Pairwise Similarity Measure

✓ Groups of nodes sharing similar neighborhood patterns in the graph

$$S = \sum_{\ell=1}^{\infty} \beta^{2(\ell-1)} \Gamma_A^{\ell}[I],$$

$$\Gamma_A : \mathbb{R}^{n\times n} \rightarrow \mathbb{R}^{n\times n} : \Gamma_A[X] = AXA^T + A^T XA$$

$$S_{k+1} = \Gamma_A[I] + \cdots + (\beta^2)^k \Gamma_A^{k+1}[I] + (\beta^2)^{k+1} \Gamma_A^{k+1}[S_0]$$

$$S_{k+1} = \Gamma_A\left[I + \beta^2 S_k\right]$$

# Pairwise Similarity Measure

$$S_{k+1} = \Gamma_A[I] + \cdots + (\beta^2)^k \Gamma_A^{k+1}[I] + (\beta^2)^{k+1} \Gamma_A^{k+1}[S_0]$$

Converges if $\rho\left(\beta^2 \Gamma_A[.]\right) < 1$

$$\Gamma_A[X] = AXA^T + A^TXA \quad \rightarrow \quad vec(\Gamma_A[X]) = \left(A \otimes A + A^T \otimes A^T\right) vec(X)$$

$$\rho\left(\beta^2\left(A \otimes A + A^T \otimes A^T\right)\right) < 1$$

$$\beta^2 < \frac{1}{\rho\left(A \otimes A + A^T \otimes A^T\right)}$$

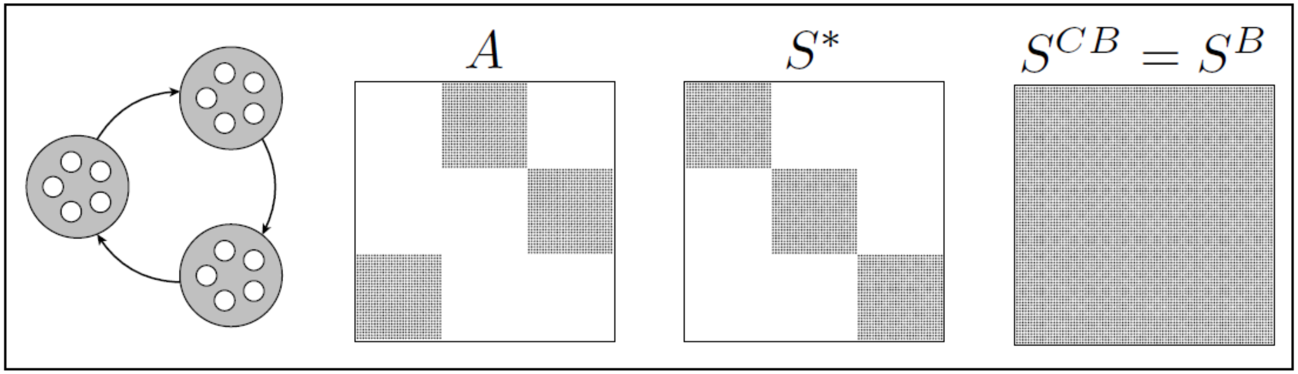
Sufficient condition : $\beta^2 \leq \frac{1}{\rho\left(A + A^T\right)^2}$

# Pairwise Similarity Measure
# Fixed point solution

$$S_0 = 0 \qquad\qquad S_1 = AA^T + A^T A$$

$$S_{k+1} = S_1 + \beta^2 \Gamma_A [S_k]$$

$$vec(S^*) = \left[I - \beta^2 \left(A \otimes A + (A \otimes A)^T\right)\right]^{-1} vec(S_1)$$




- × Exact solution: intractable
- × Power method: expensive $O(n^3)$

# Low rank Approximation

$$S_k^{(r)} = X_k X_k^T \,, \qquad X_k \in \mathbb{R}^{n\times r}$$

$$S_{k+1}^{(r)} = \Pi^{(r)}\left[ S_1^{(r)} + \beta^2 \Gamma_A \left[ S_k^{(r)} \right] \right] = X_{k+1}\, X_{k+1}^T$$

Rank $3r$

$$S_1 = A\,A^T + A^T\,A = \left[A \mid A^T\right]\left[A \mid A^T\right]^T$$

Truncated SVD $\left[A \mid A^T\right] \approx U_1 \Sigma_1 V_1^T$

$$\begin{aligned} S_1^{(r)} &= \Pi^{(r)}\left[ \left[A \mid A^T\right]\left[A \mid A^T\right]^T \right] \\ &= U_1 \Sigma_1^2 U_1^T = X_1 X_1^T \end{aligned} \qquad X_1 = U_1 \Sigma_1$$

# Low rank Approximation

## Iterative solutions

$$S_{k+1}^{(r)} = \Pi^{(r)} \left[ S_1^{(r)} + \beta^2 \Gamma_A \left[ S_k^{(r)} \right] \right] = X_{k+1} \, X_{k+1}^T$$

$$S_1^{(r)} + \beta^2 \Gamma_A \left[ S_k^{(r)} \right] = X_1 X_1^T + \beta^2 A X_k X_k^T A^T + \beta^2 A^T X_k X_k^T A$$

$$= Y_k \, Y_k^T$$

$$Y_k = \left[ X_1 \mid \beta A X_k \mid \beta A^T X_k \right]$$

# Low rank Approximation
## Iterative solutions

$$X_{k+1}X_{k+1}^T = \Pi^{(r)}\left[Y_k Y_k^T\right]$$

$$Y_k = \left[X_1 \mid \beta A X_k \mid \beta A^T X_k\right]$$

QR factorization $\quad Y_k = Q_k R_k$

(keep the first r columns of $Q_k$ )

$$X_{k+1} = Q_k \mathcal{U}_k \Omega_k$$

Truncated SVD $\quad R_k \approx \mathcal{U}_k \Omega_k \mathcal{V}_k$

Existence of Fixed Point solution and Guaranteed
local convergence of the sequence for sufficiently small $\beta$ !

Stewart, Error and perturbation bounds for subspaces associated with certain eigenvalue problems [1973]

# Low rank Projection
## Convergence

$\Delta$ small symmetric perturbation and $S^{(r)}$ low rank fixed point solution

$$f(S) = S_1^{(r)} + \beta^2 \Gamma_A [S] \qquad S^{(r)} = \Pi^{(r)} \left( f \left( S^{(r)} \right) \right) \qquad S^{(r)} = U \Sigma^2 U^T$$

$$[U\ V]^T \, f(S^{(r)}) \, [U\ V] = \begin{bmatrix} \Sigma^2 & \\ & \sigma^2 \end{bmatrix}$$

$$f(S^{(r)} + \Delta) = f(S^{(r)}) + \beta^2 \Gamma_A [\Delta]$$

$$[U\ V]^T \, \left( f(S^{(r)}) + \beta^2 \Gamma[\Delta] \right) \, [U\ V] = \begin{bmatrix} E_{11} & E_{21}^T \\ E_{21} & E_{22} \end{bmatrix}$$

# Low rank Projection

## Convergence

$$[U\ V]^T \left(f(S^{(r)}) + \beta^2\Gamma[\Delta]\right) [U\ V] = \begin{bmatrix} E_{11} & E_{21}^T \\ E_{21} & E_{22} \end{bmatrix}$$

There exists $Q \in \mathbb{R}^{n\times r}$ such that $UQ$ is an invariant subspace if

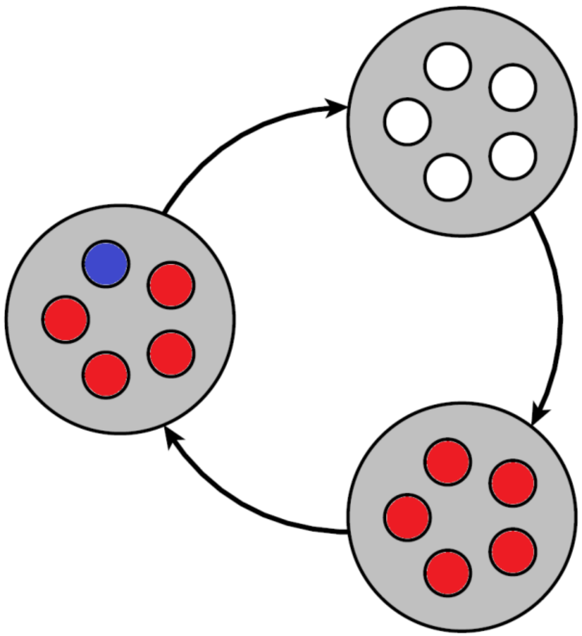
$$0 \leq 4\beta^2 \left\|\Gamma\left[\Delta\right]\right\|_F \leq \Sigma_{r,r}^2 - \sigma_{1,1}^2$$

It implies that $\left\|S^{(r)} - \Pi^{(r)}\left[f(S^{(r)} + \Delta)\right]\right\|_F \leq \gamma \left\|\Delta\right\|_F$

$\gamma < 1$ if $\beta^2 < \dfrac{1}{\|A\otimes A + A^T \otimes A^T\|_2 \left(\frac{4\|\Sigma^2\|}{\Sigma_{r,r}^2 - \sigma_{1,1}^2} + 1\right)}$

# Erdos-Reyni random graphs
## with a block stucture

$$G_B(V_B, E_B)$$



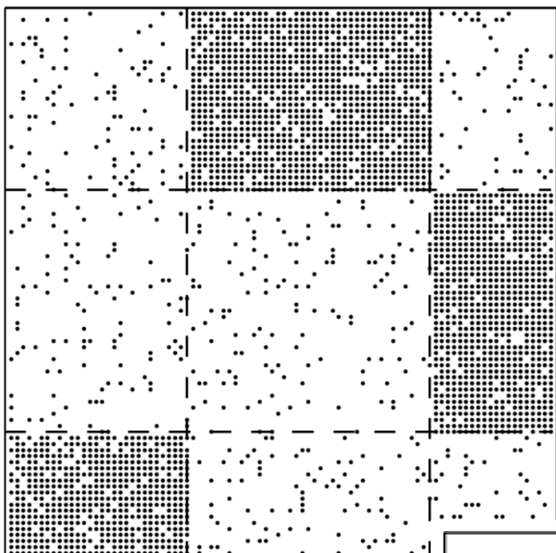
$$i, j \in V_A$$

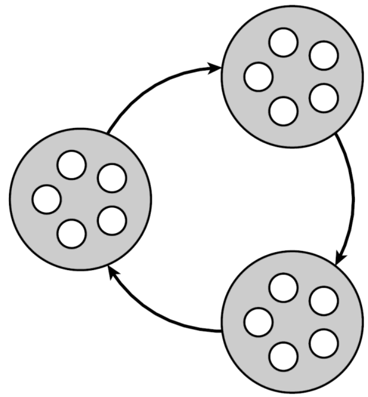
$$(i, j) \in E_A \text{ w.p. } p_{out}$$

$$\text{if } (R(i), R(j)) \notin E_B$$

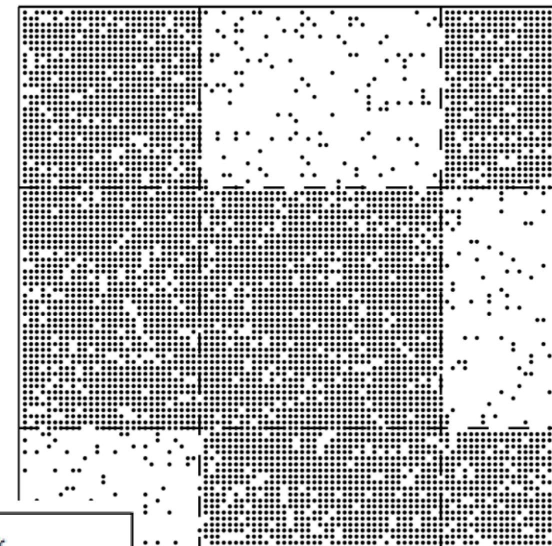
# Erdos-Reyni random graphs
## with a block stucture

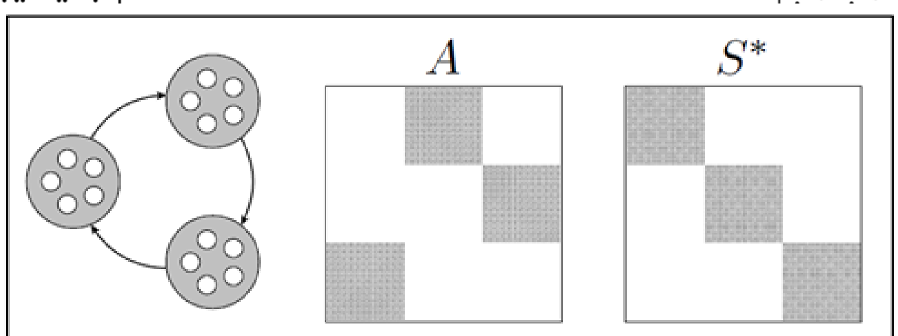
$p_{in} >> p_{out}$





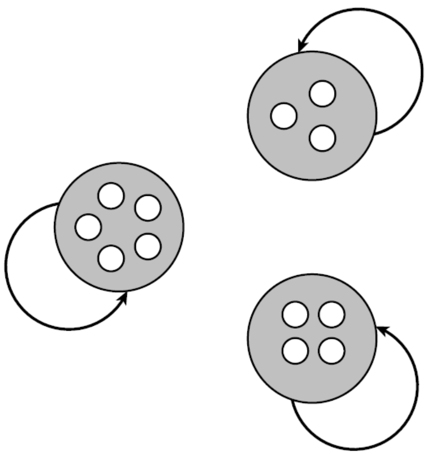
$p_{in} << p_{out}$

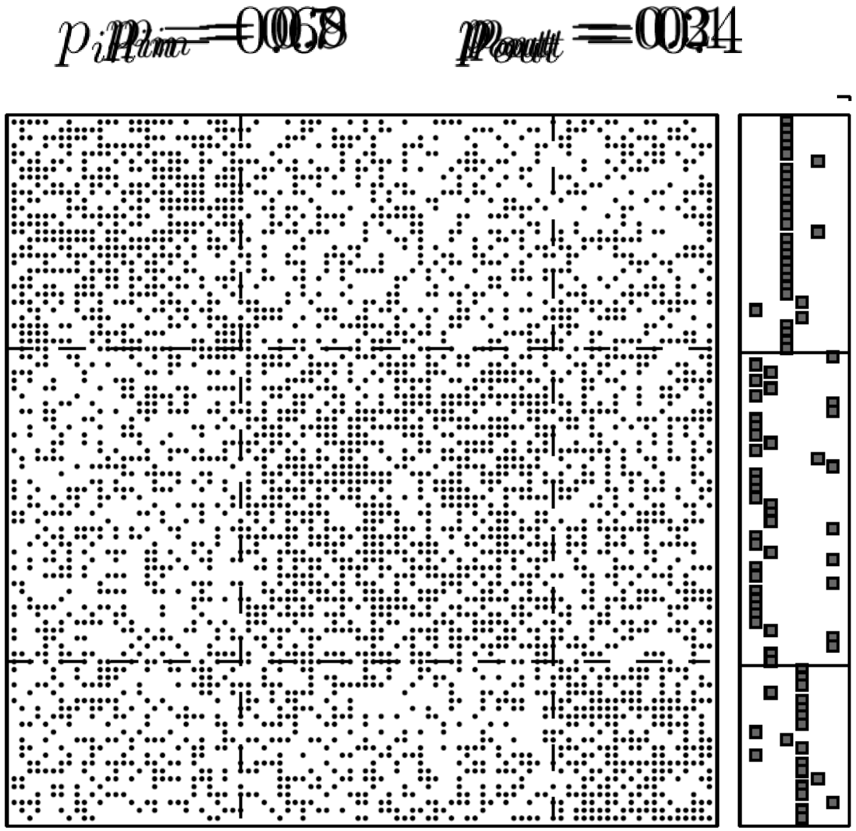




# Erdos-Reyni random graphs

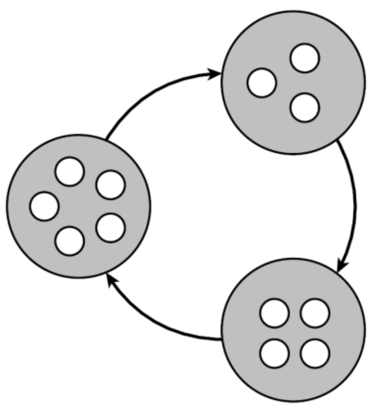
## Results

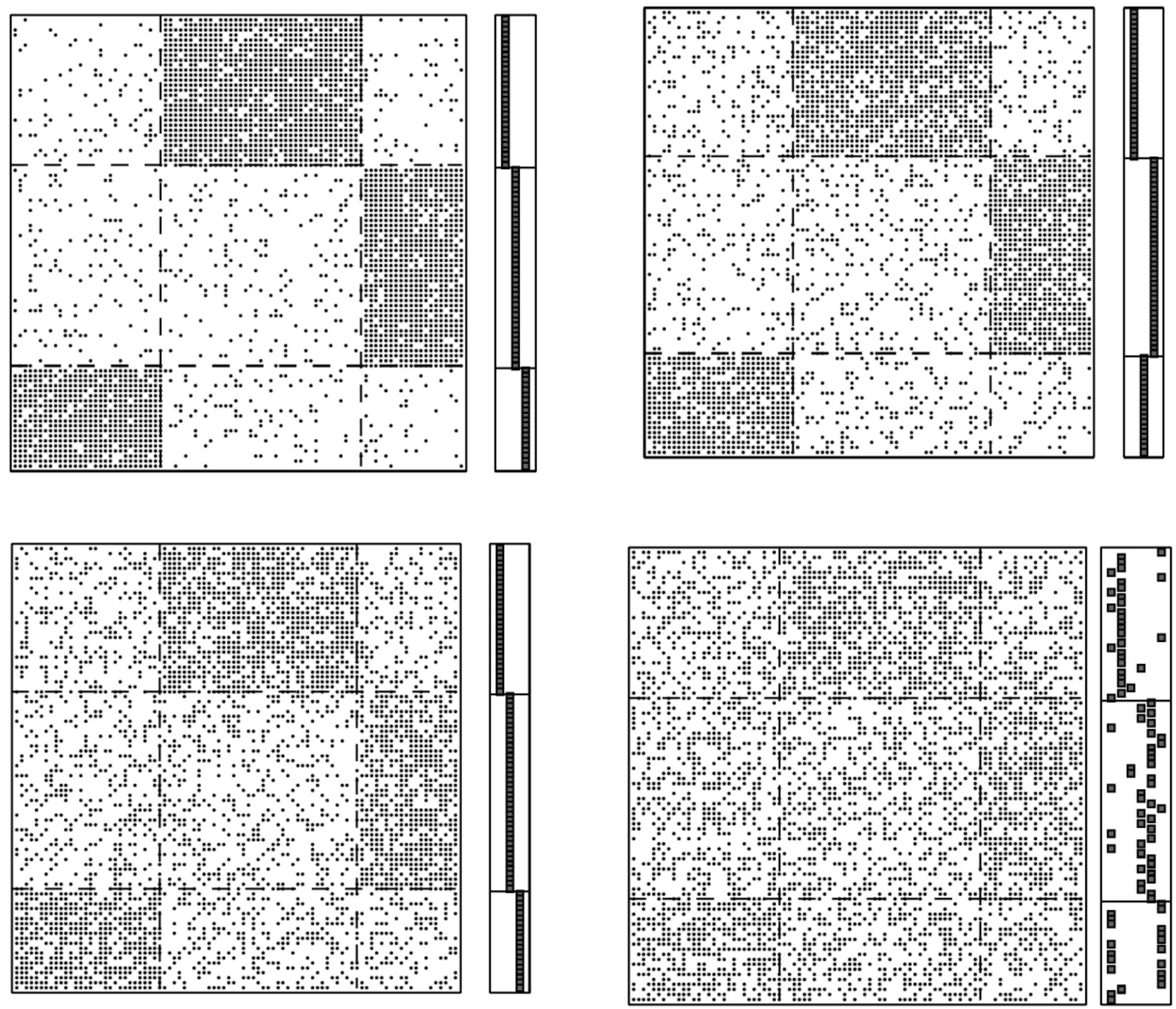




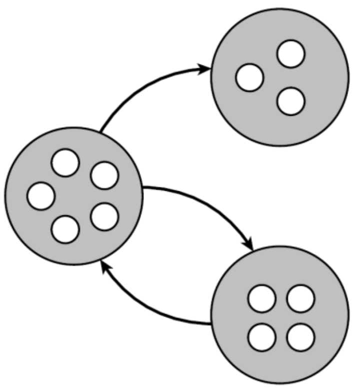
# Erdos-Reyni random graphs

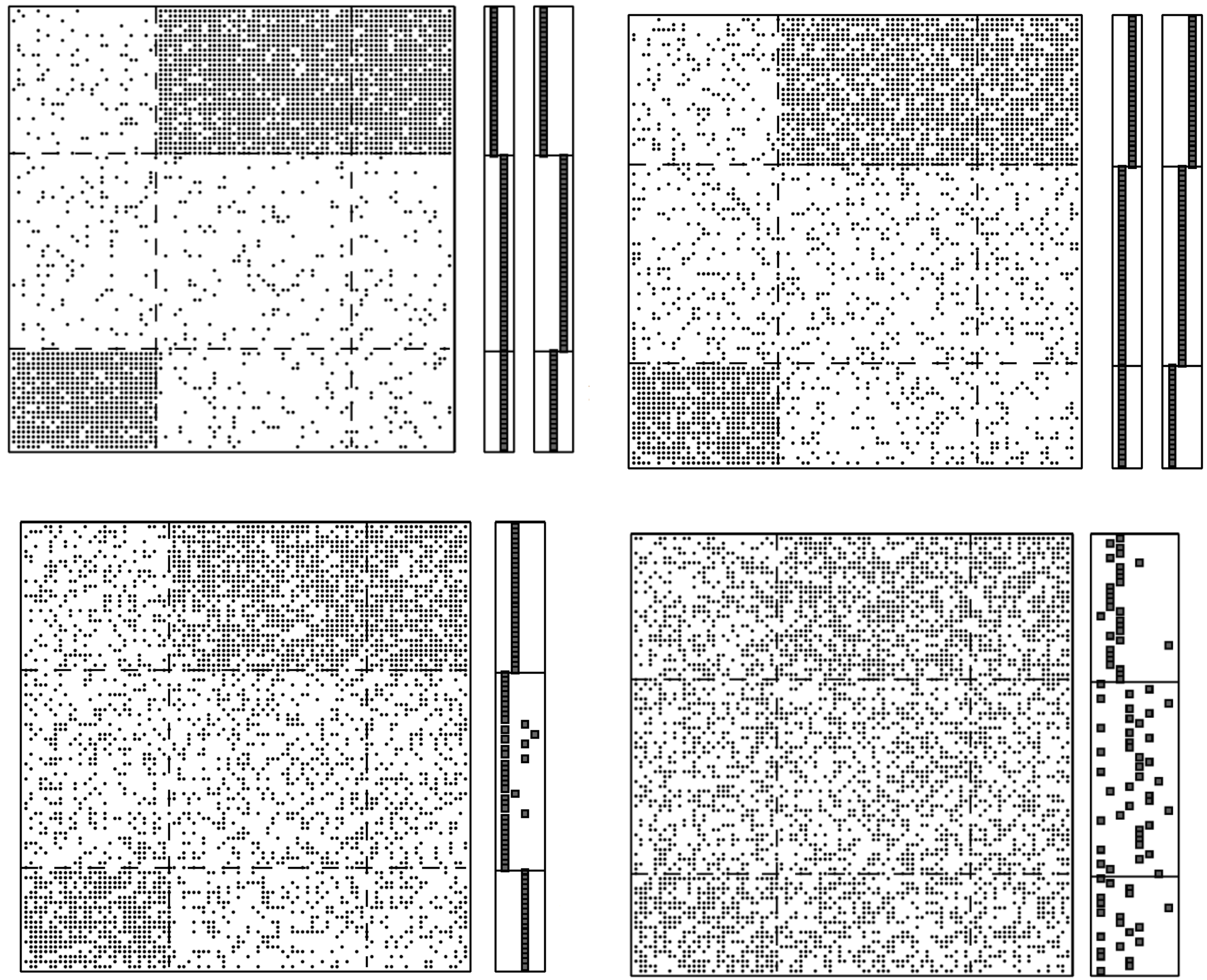
## Results





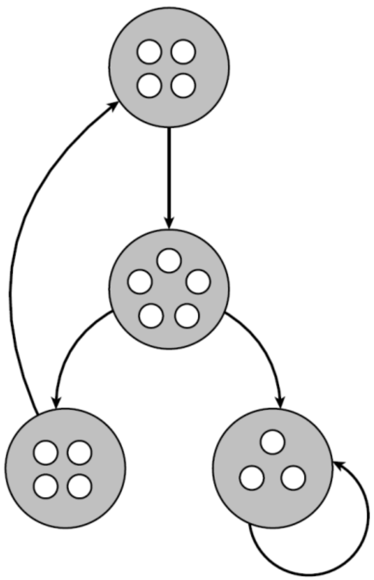
# Erdos-Reyni random graphs
## Results

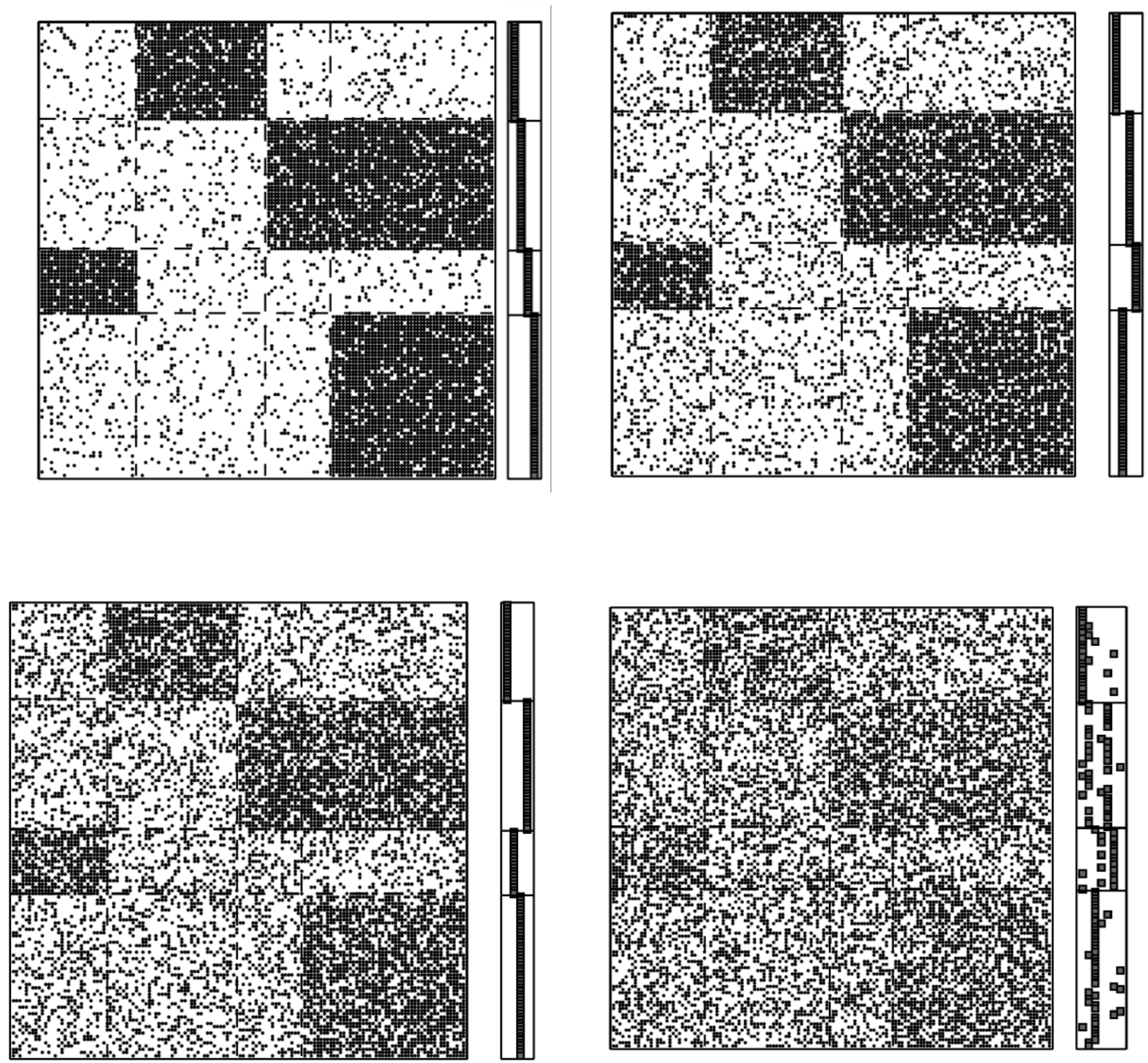




# Erdos-Reyni random graphs

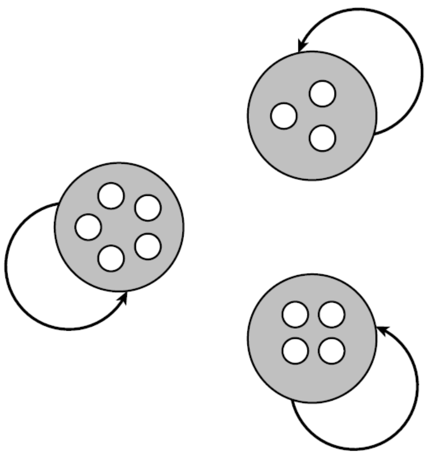
## Results

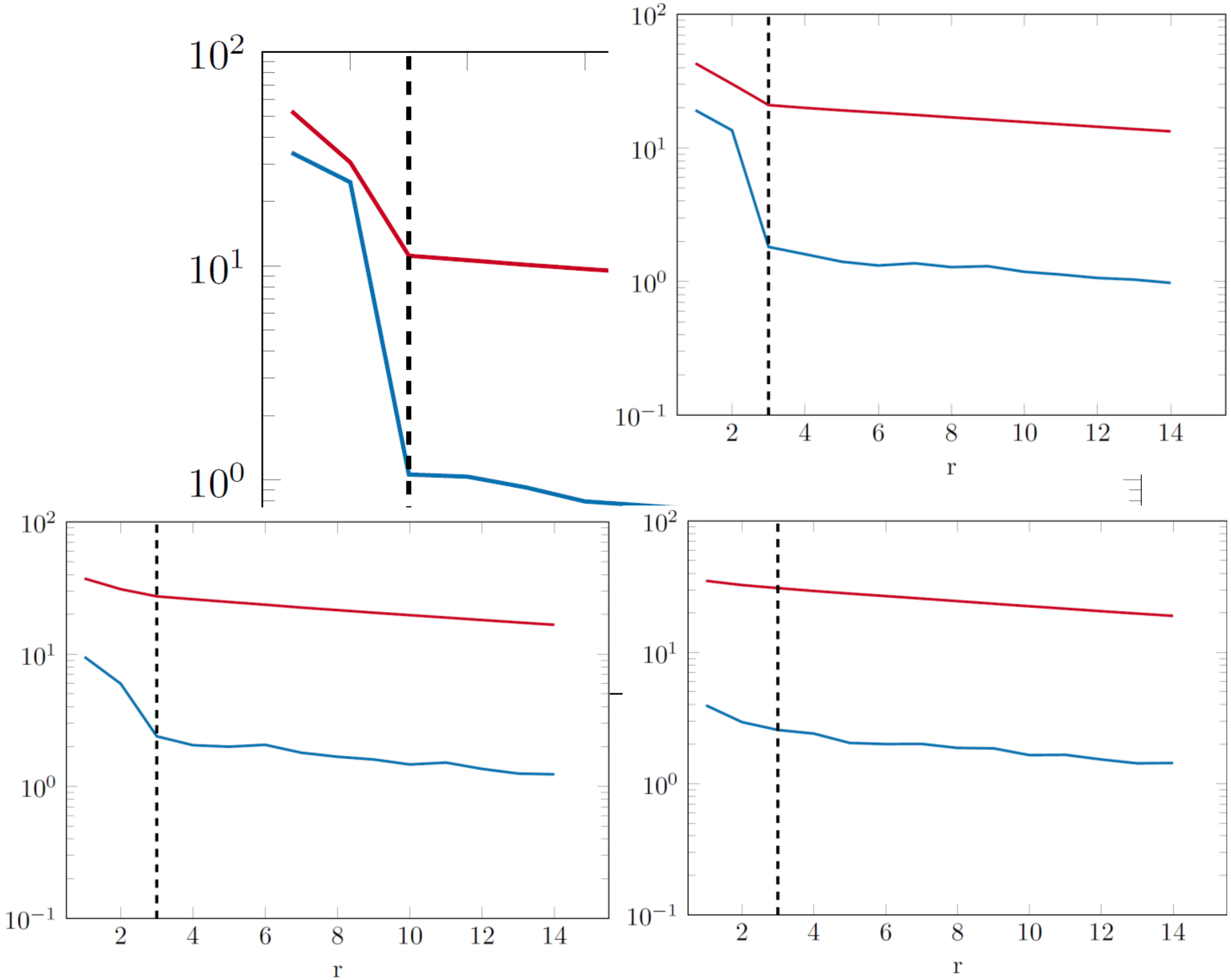




# Erdos-Reyni random graphs

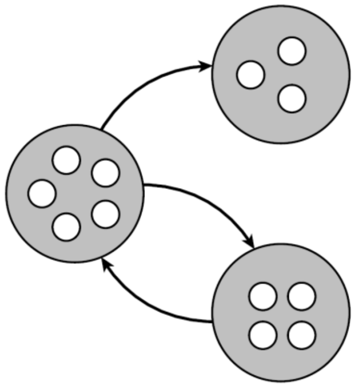
## Results

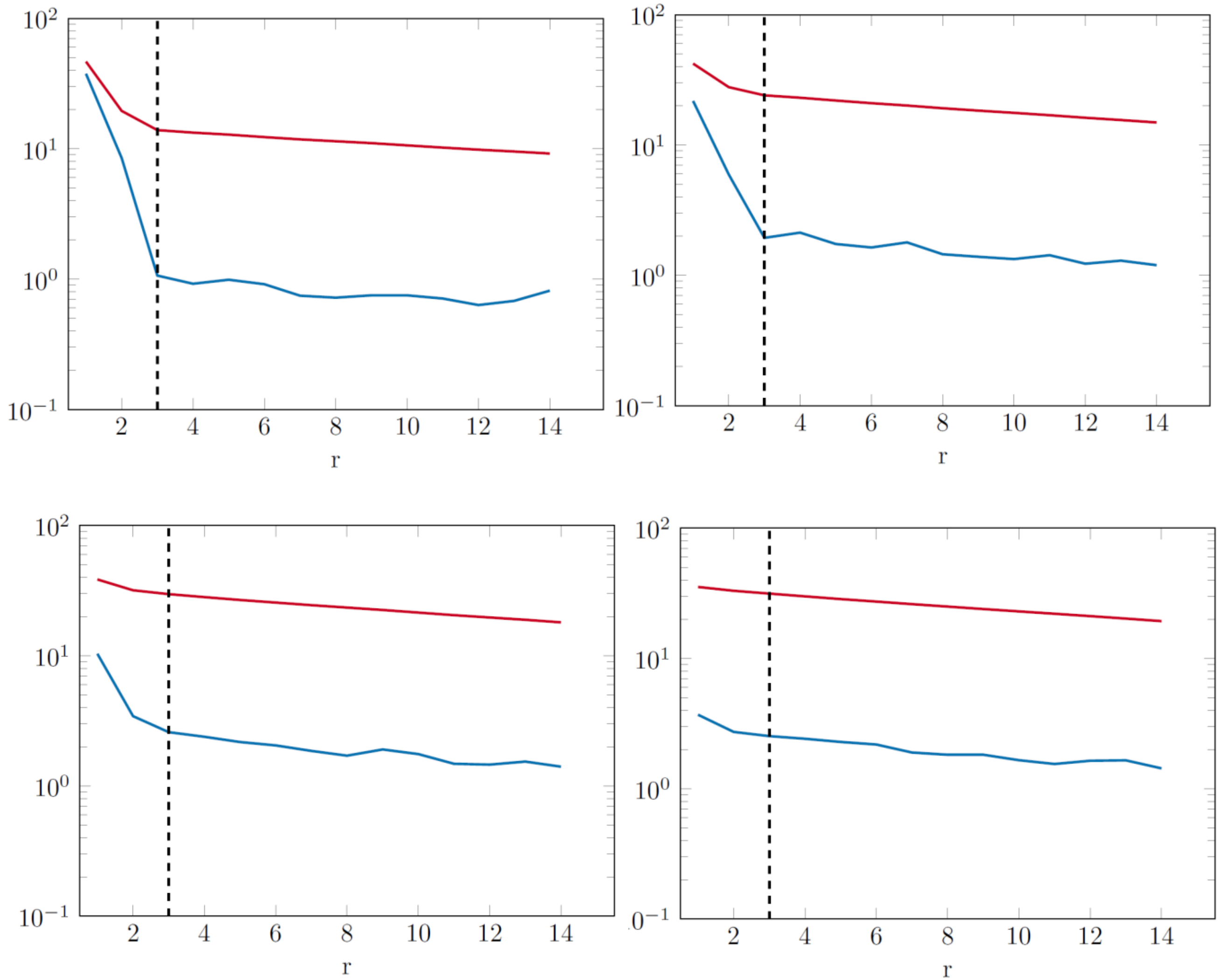




# Erdos-Reyni random graphs

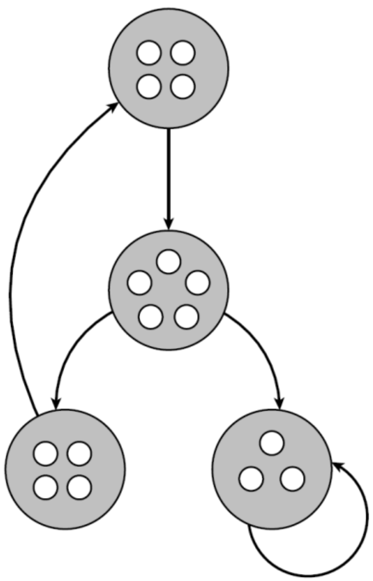
## Results

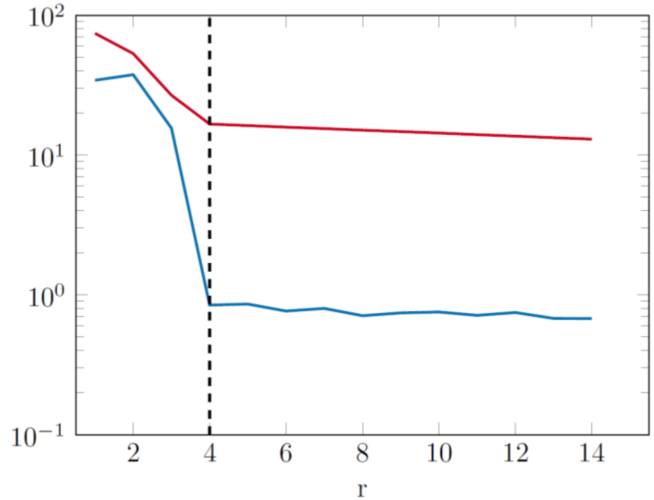


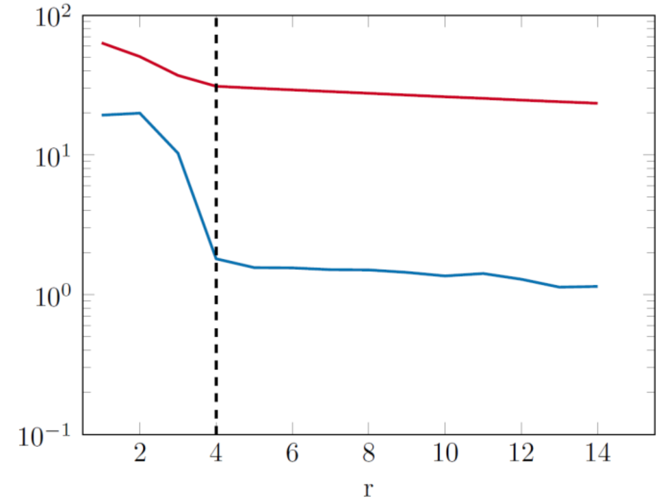


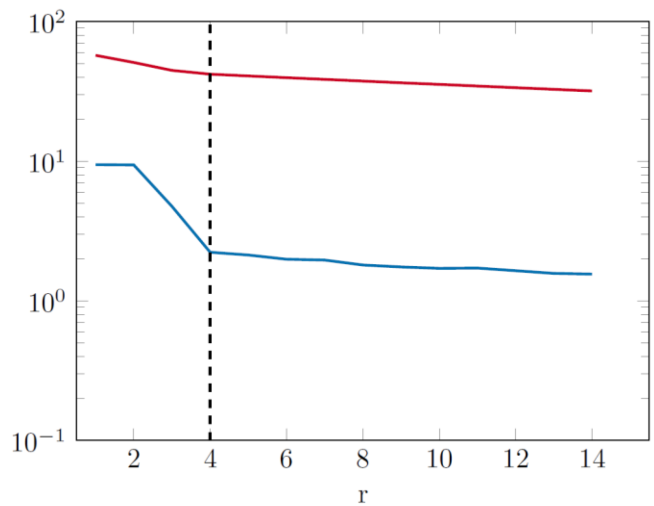
# Erdos-Reyni random graphs

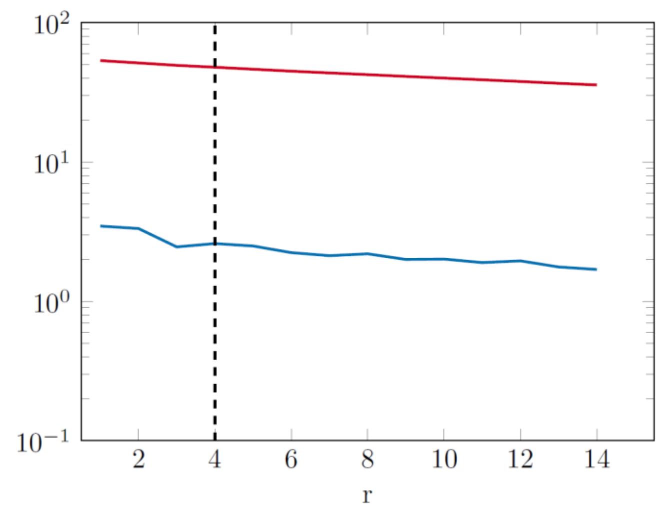
## Results





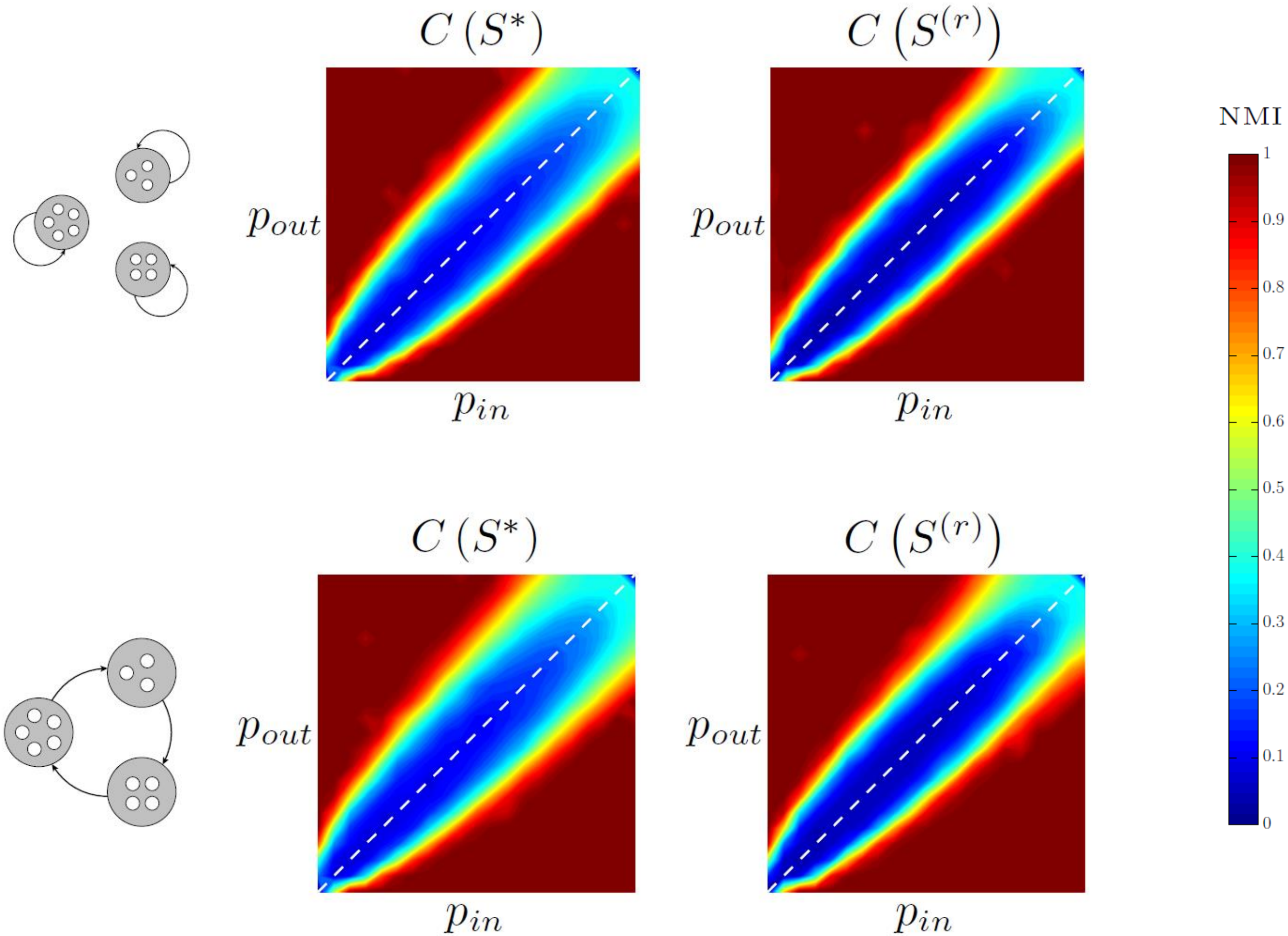



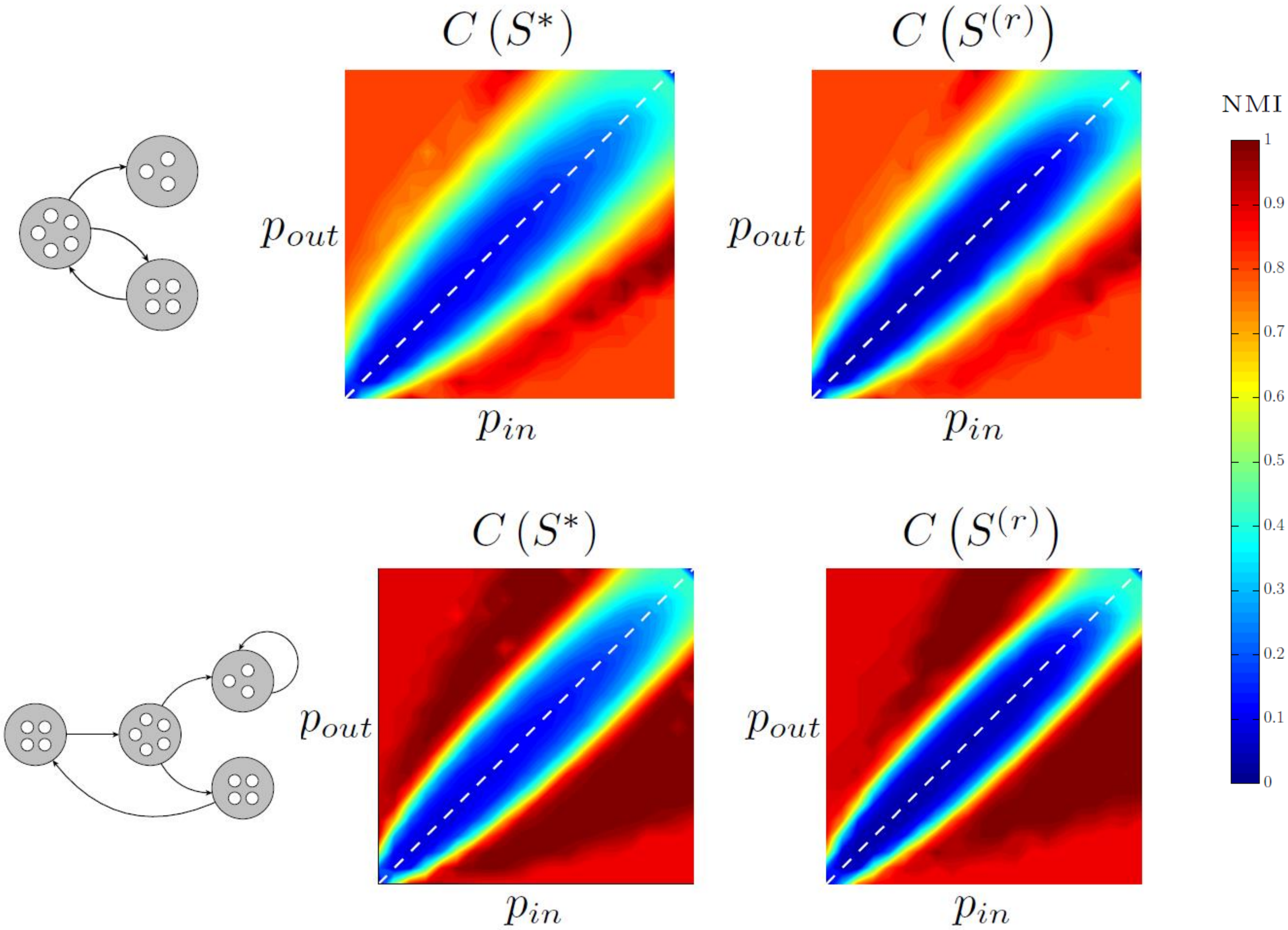



# Erdos-Reyni random graphs

## Results

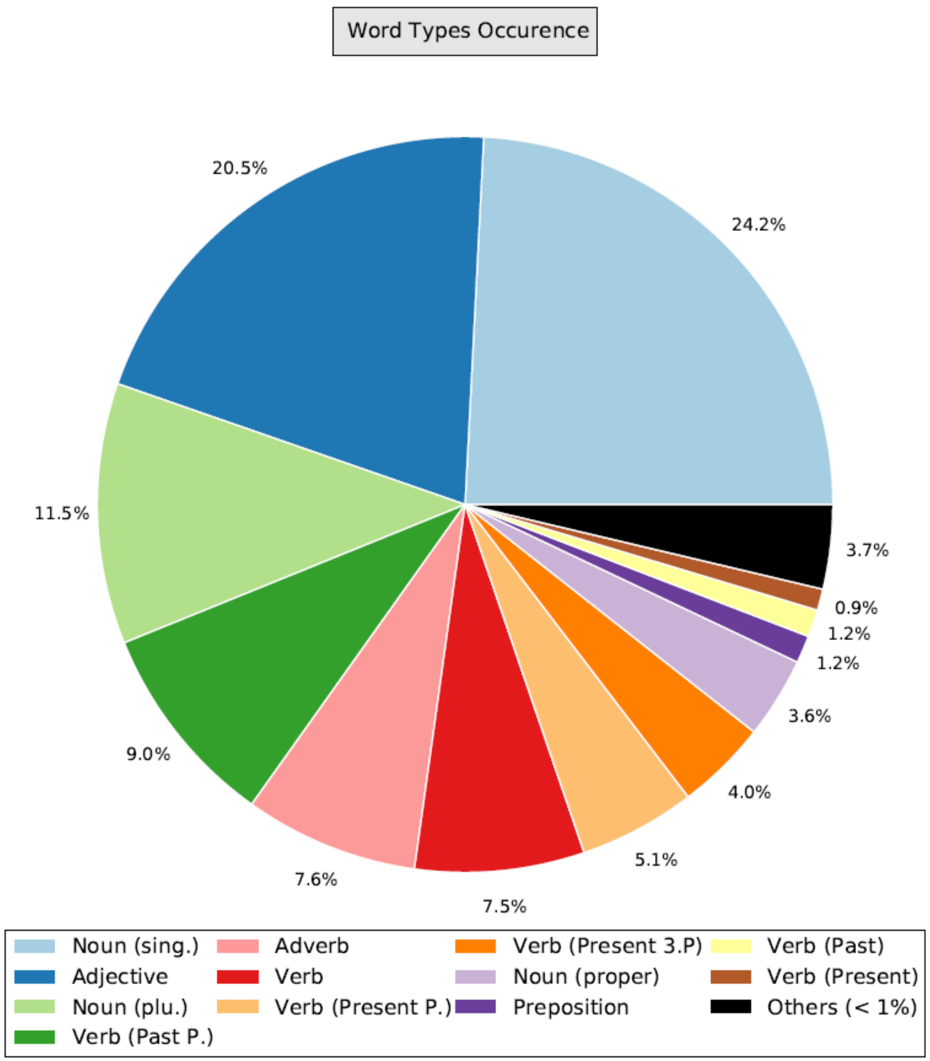


# Erdos-Reyni random graphs
## Results



# Words graph

« Language, an introduction to the study of speech » *Sapir. (1972)*

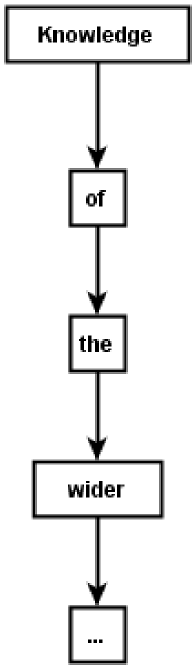



Noun (singular)

Preposition

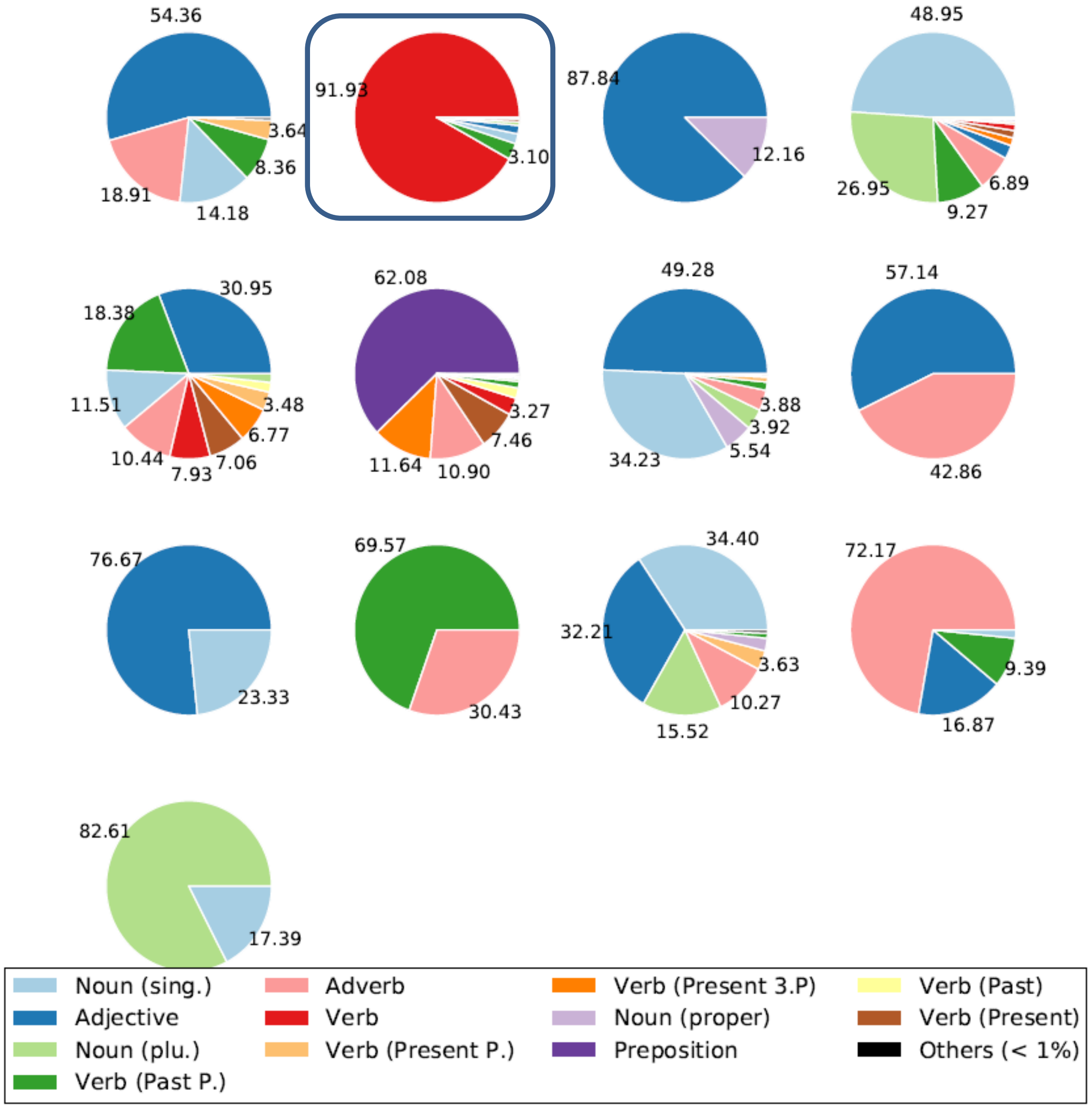
Determinant

Adjective



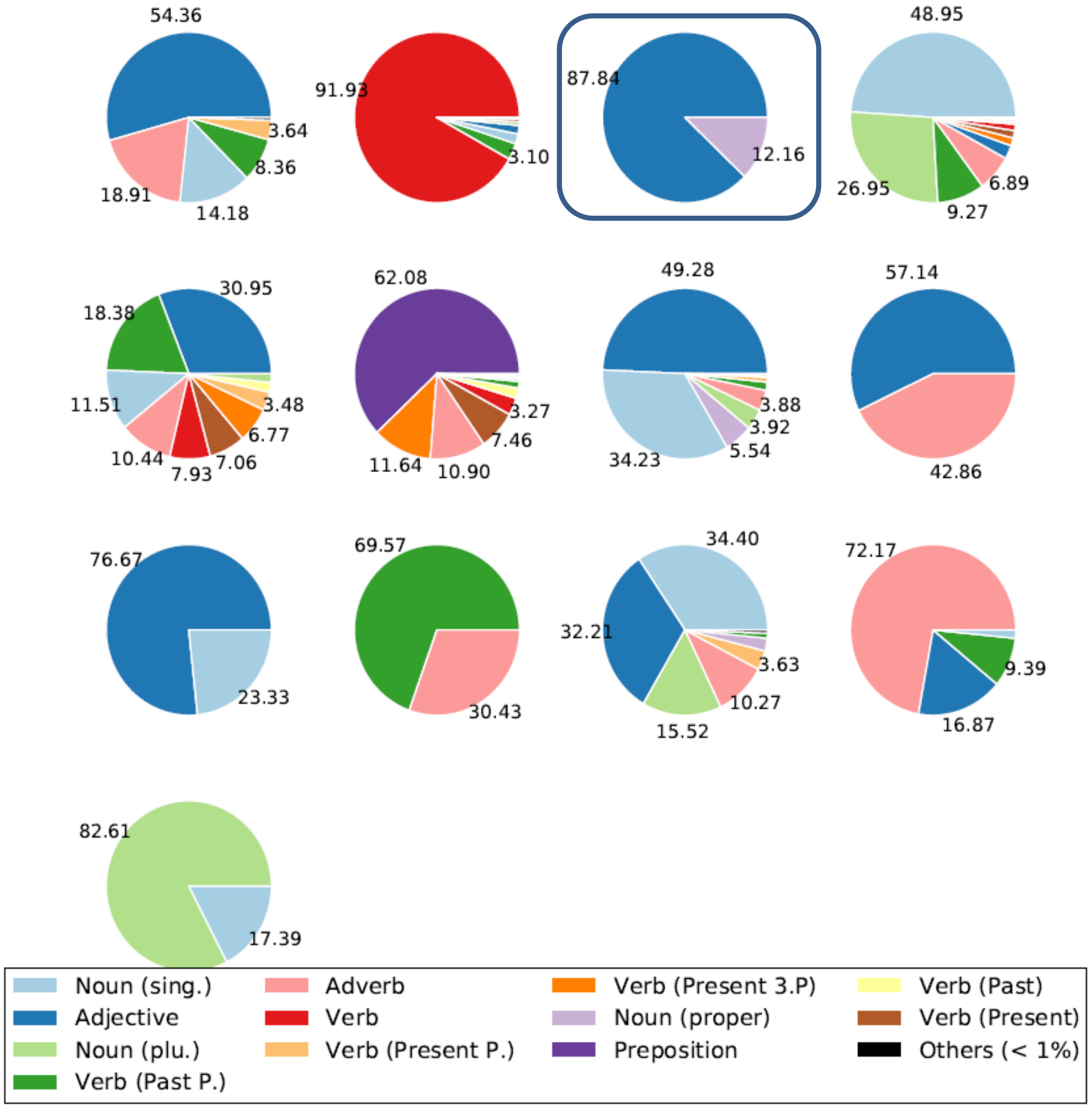

*Python lib: NLTK*
*+ Stanford PoS Tagger*

# Words graph



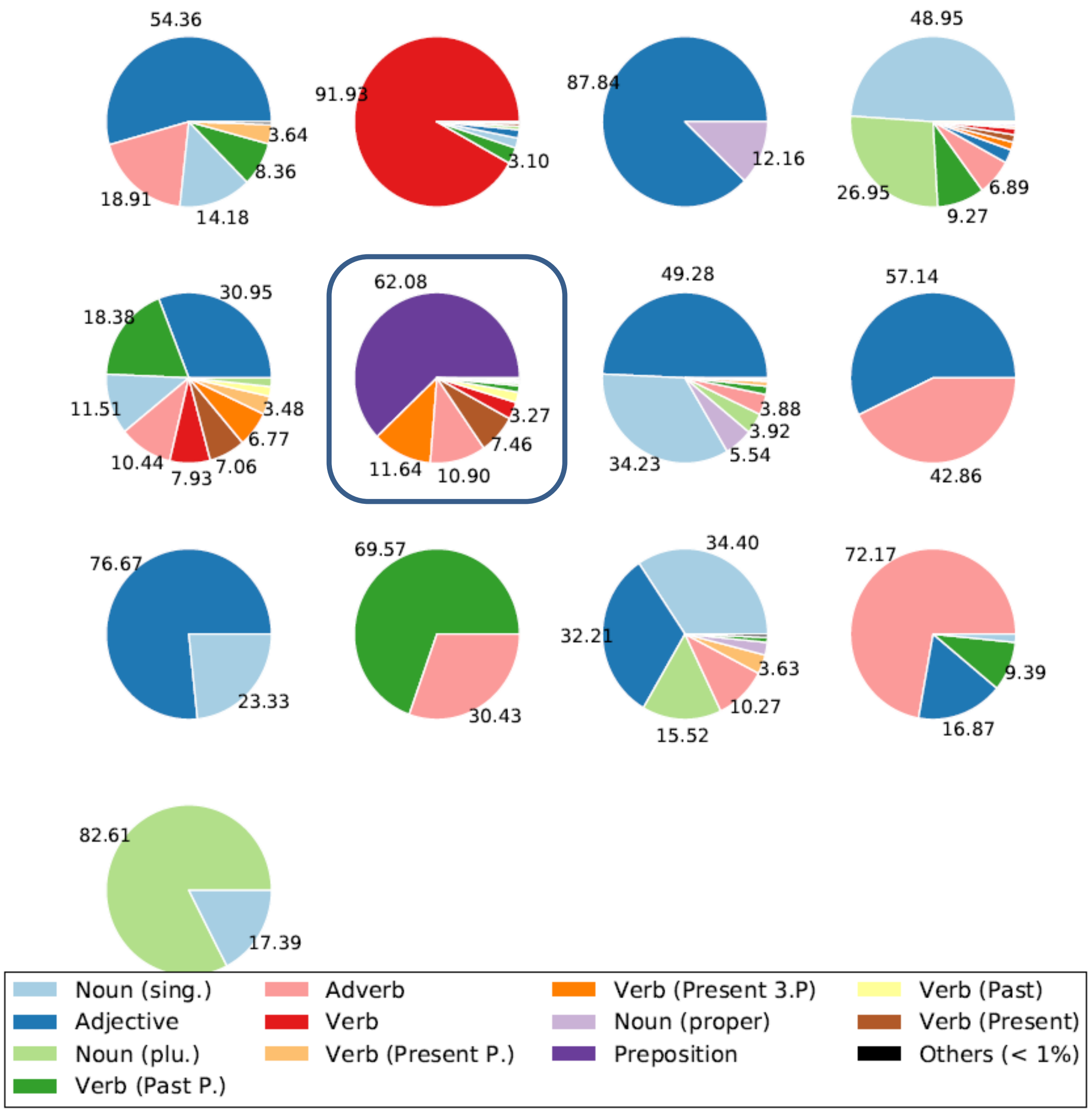

| Cluster 2 |
|---|
| be (582) |
| say (128) |
| go (37) |
| become (33) |
| him (28) |
| show (24) |
| believe (23) |
| me (17) |
| animate (17) |
| indicate (16) |
| look (15) |
| fall (14) |
| observe (14) |
| serve (12) |
| run (12) |

# Words graph



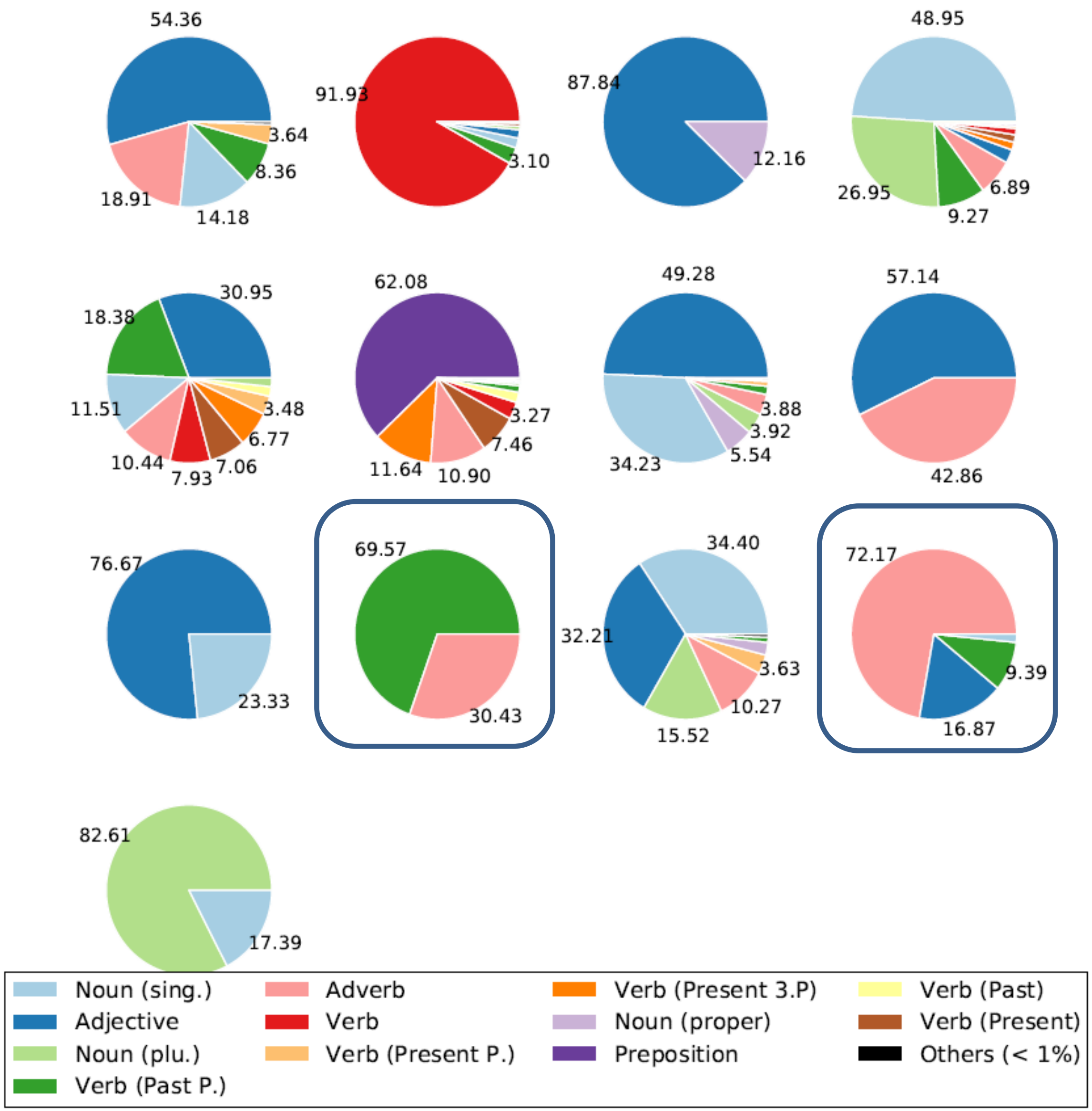

| Cluster 3 |
|---|
| greek (48) |
| elusive (10) |
| cambodgian (9) |
| archaic (7) |
| satisfying (5) |
| grouped (4) |
| siamese (4) |
| nowhere (4) |
| inclusive (4) |
| explicit (4) |
| religious (4) |
| infixes (4) |
| treated (4) |
| formless (4) |
| syllabic (3) |

# Words graph




| Cluster 6 |
|---|
| of (3636) |
| to (1820) |
| in (1647) |
| is (1557) |
| and (1514) |
| that (1220) |
| as (1100) |
| or (795) |
| are (766) |
| not (607) |
| by (411) |
| with (391) |
| but (387) |
| for (366) |
| have (364) |

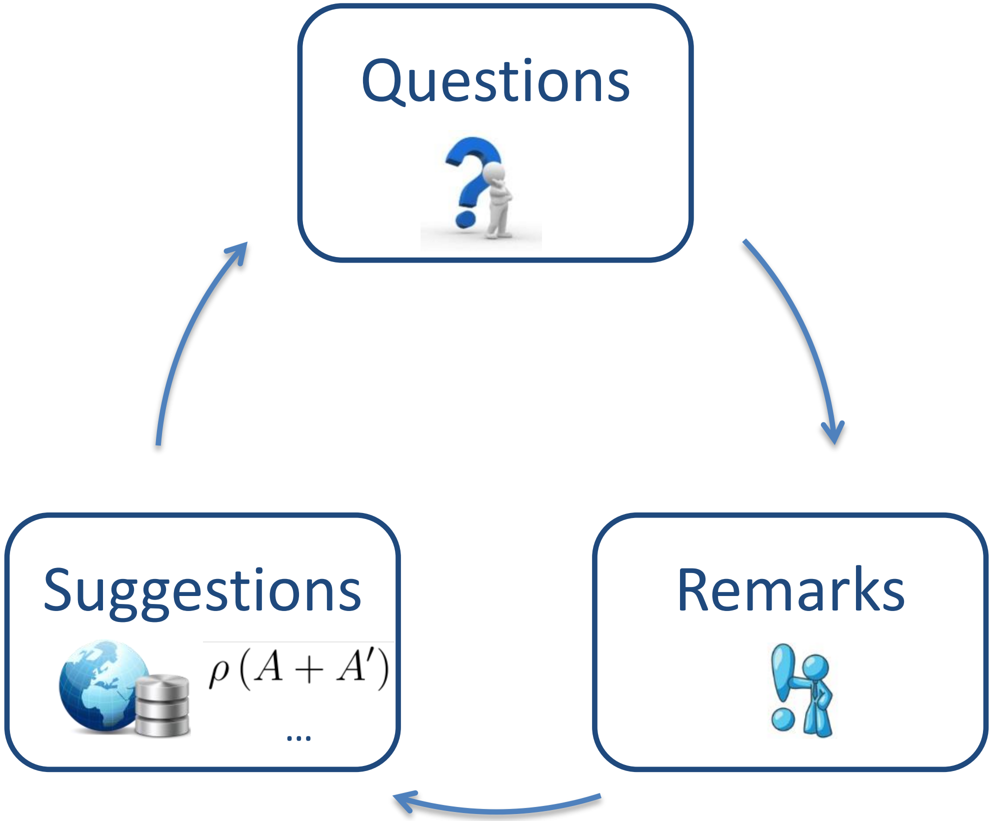
# Words graph




| Cluster 10 | Cluster 12 |
|---|---|
| looked (16) | only (137) |
| conveniently (7) | still (64) |
| enormously (4) | too (60) |
| rapidly (4) | entirely (34) |
| eliminated (4) | never (33) |
| termed (4) | hardly (31) |
| barely (4) | identical (25) |
| obsolete (3) | really (22) |
| sleeping (3) | clear (16) |
| swept (2) | concerned (15) |
| subdivided (2) | red (13) |
| multiplied (2) | obvious (12) |
| diffused (2) | indicated (11) |
| tired (2) | constantly (10) |
| abundantly (1) | done (10) |

# Take Home

- ✓ Efficient and highly parallelizable algorithm for community detection
- ✓ Role Extraction or Block Modeling generalized community detection
- ✓ The pairwise node similarity measure allows to extract such roles
- ✓ Accurate low rank approximation for large graphs


Questions
Remarks
Suggestions
$\rho(A + A')$
...