Optimizing a Hierarchical Partition of a Complex Network

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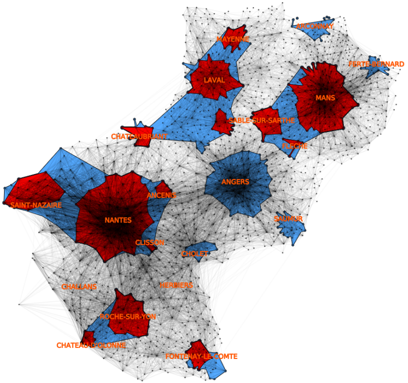
Hierarchical community structure

A partition of the vertices where each cluster can be recursively subdivided into a new partition.

Example : *Pays-de-la-Loire* commuters network (1999) with the *ZAUER* classification (INSEE).



State of the art (algorithms producing or *using* a hierarchical partition)

- **Agglomerative:**
  - Hierarchical Clustering based on distances [Ward Jr, 1963]
  - Iterative call of a "flat" clustering algorithm the compound graph induced by the previous partition [Blondel et al., 2008]
- **Divisive:**
  - Edge filtering [Girvan and Newman, 2002, Radicchi et al., 2004]
  - Iterative call of a "flat" clustering algorithm on each previously detected cluster [Auber et al., 2003]
- **Others:**
  - Probabilistic models [Clauset et al., 2006]
  - *Hierarchical Infomap* [Rosvall and Bergstrom, 2011] (hybrid method)
  - Dendrogram filtering using a scale parameter [Pons and Latapy, 2011]

Search space size

For $i = 1 \ldots 10$

- $\#\{$**Flat Partition**$\}$ **:** (*Bell number*)
  1, 2, 5, 15, 52, 203, 877, 4140, 21147
- $\#\{$**Dendrograms**$\}$ **:** (*Double factorial*)
  1, 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425
- $\#\{$**Hierarchical Partitions**$\} \geq$ ([Flajolet and Sedgewick, 2009])
  1, 1, 4, 26, 236, 2752, 39208, 660032, 12818912, 282137824

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Problems:

- Evaluate the quality of a hierarchical partition of a graph.
- Exploring the space of hierarchical partitions

Hierarchical Partition

Let $G = (V, E)$ be a graph, a hierarchical partition is a (partition) tree T where :

- Each node $t \in T$ corresponds to a subset $C_t \subset V$.
- σ_t is the list of successors of t and we have $C_t = \bigcup_{f \in \sigma_t} C_f$.
- $\mathcal{F}(T)$ (the *leaves*) is the subset of nodes having $\sigma_t = \emptyset$.
- $N_i(T)$ is the ith level of T *i.e.* the set of nodes at a distance **at most** i from the root such as $\bigcup_{t \in N_i(T)} C_t = V$.
- T_t (resp. G_t) is the subtree rooted in t (resp. the subgraph induced by C_t).

Remark : A flat partition $\mathcal{C}$ is a hierarchical partition with $N_1(\mathcal{C}) = \mathcal{F}(\mathcal{C})$.
Example (Commuters networks) : Classify the cities according to the administrative region and then according to the "departement".

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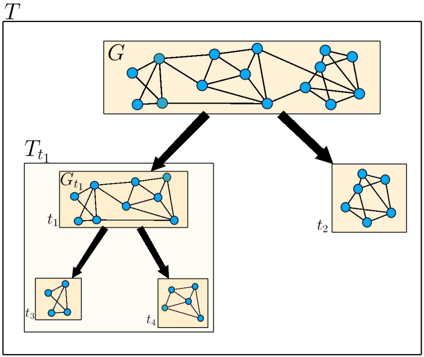

- $\sigma_{t_1} = \{t_3, t_4\}$
- $N_1(T) = \{t_1, t_2\}$
- $\mathcal{F}(T) = N_2(T) = \{t_3, t_4, t_2\}$

Quality measure

A *quality measure* is a function $\Phi(G, \mathcal{C}) \rightarrow \mathbb{R}$.
If $\Phi(G, A) > \Phi(G, B)$, A is said "to be better" than B.

Examples:

- Modularity Q [Newman, 2006]
- Mancoridis index MQ [Mancoridis et al., 1998]
- Performance [van Dongen, 2000]
- Map Equation [Rosvall and Bergstrom, 2008]
- Surprise [Aldecoa and Marín, 2011]

Additive quality measure

An *additive quality measure* is a quality measure which can be written:

$$\Phi(G, \mathcal{C}) = \sum_{i}^{k} \phi(G, \mathcal{C}, C_i)$$

where $\phi(G, \mathcal{C}, C_i)$ is the *gain* of the cluster C_i.

Examples:

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- ~~Surprise [Aldecoa and Marín, 2011]~~

Strongly additive quality measure [Pons and Latapy, 2011]

A *strongly additive quality measure* is a quality measure which can be written:

$$\Phi(G,\mathcal{C}) = \sum_{i}^{k} \phi(G, C_i)$$

where $\phi(G, C_i)$ is the *gain* of the cluster C_i
(does not depend on the way $(V \setminus C_i)$ is partitioned).

Examples:

- Modularity Q [Newman, 2006]
- Mancoridis index MQ [Mancoridis et al., 1998] (weighted version)
- Performance [van Dongen, 2000]
- ~~Map Equation [Rosvall and Bergstrom, 2008]~~
- ~~Surprise [Aldecoa and Marín, 2011]~~

Hierarchical quality measure [Queyroi et al., 2011]

Let $G = (V, E)$ be a graph with a hierarchical partition T and ϕ be the gain function for a cluster. The *hierarchical quality measure* is

$$\Phi(G, T, q) = \sum_{i \in \sigma_{r(T)}} \phi(G, C_i)\,(1 + q \times \Phi(G_i, T_i, q))$$

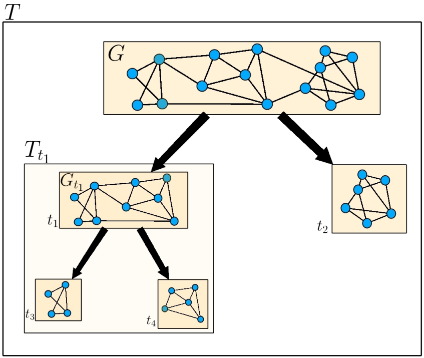
with $q \in [0, 1]$. $r(T)$ is the root of T.

Remarks :

- $\Phi(G, T, q)$ is a one-variable polynomial.
- The variable q can be used to favour deeper hierarchical clustering.

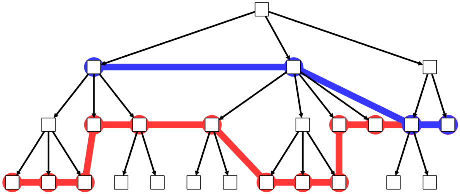
Without loss of generality, we will use as hierarchical quality criterion:

$$\Phi(G, T) = \int_0^1 \Phi(G, T, q)dq$$



$$\Phi(G, T, q) = \phi(G, C_{t_1}) + \phi(G, C_{t_2}) + q\ \phi(G, C_{t_1})\,(\phi(G_{t_1}, C_{t_3}) + \phi(G_{t_1}, C_{t_4}))$$
$$\Phi(G, T) = \phi(G, C_{t_1}) + \phi(G, C_{t_2}) + \frac{1}{2}\ \phi(G, C_{t_1})\,(\phi(G_{t_1}, C_{t_3}) + \phi(G_{t_1}, C_{t_4}))$$

Objective: Find the best sub-hierarchical partition induced by a hierarchy



Equivalent to

- Find the best subset of (non-horizontal) cuts
- Find the best subset of nodes of T to remove

Number of solutions is $\mathcal{O}(2^{|T|})$ $\Rightarrow$ **Use a heuristic**

The gain of node removal

Let $t \in T$, The gain of the removal of t is the difference between the quality of T and the quality of $T \setminus \{t\}$.

$$\Delta_t \Phi(G, T) = \Phi(G, T \setminus \{t\}) - \Phi(G, T)$$

The removal of a internal node t has several effects:

- The nodes of T_t are now "higher" in the hierarchy.
- σ_t is no longer a partition of G_t but is part of a partition of $G_{a(t)}$ (where $a(t)$ is the direct ancestor of t).

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Algorithm 1: Greedy Optimisation of T.

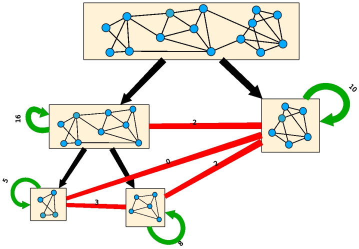
Input: $G = (V,E)$ a graph, T a hierarchical clustering of G
Output: T' a hierarchical clustering of G

```
T' = T;
t_max = arg max_{t ∈ T'} Δ_t Φ(G, T');
while Δ_{t_max} Φ(G, T) > 0 do
    T' = T' \ {t_max};
    t_max = arg max_{t ∈ T'} Δ_t Φ(G, T');
end
return T';
```

Time complexity

The time complexity of the procedure is $\mathcal{O}(h^2|E| + |T|^3)$ where $|T|$ is the number of nodes in T and h the height of T.

- The number of internal/external edges are computed in $\mathcal{O}(h^2|E|)$.
- Computation of $\phi(G, C_t)$ in constant time.
- Computation of $\Phi(G, T)$ (and $\Delta_t\Phi(G, T)$) is done using a tree traversal.



Modularity [Newman, 2006]

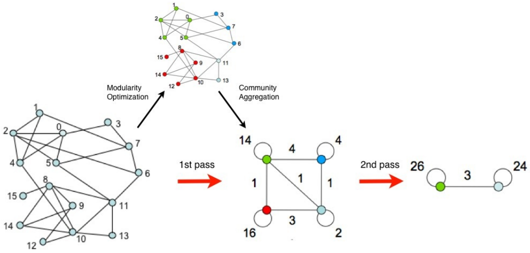
Let $G = (V, E)$ be a graph with a flat clustering $\mathcal{C}$

$$Q(G, \mathcal{C}) = \sum_{i}^{k} \frac{e_i}{|E|} - \left(\frac{d_i}{2|E|}\right)^2$$

where

- e_i : number of *internal* edges for cluster C_i
- d_i : sum of the degrees of vertices in C_i

[Blondel et al., 2008] introduced a heuristic (*Louvain* algorithm) for maximizing the modularity.



Remarks

- The iterative call leads to the construction of a hierarchy
- The output of the algorithm is the first level $N_1(T)$
- The produced hierarchical clustering can be of interest for some applications
- Some internal clusters may be irrelevant (the heuristic used may not be able to group a lot of vertices in one step)

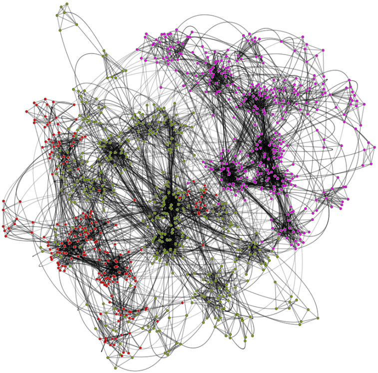
Modularity gain

We use $\forall t \in T$

$$\phi(G, C_t) = \frac{e_t}{e_{a(t)}} - \left(\frac{\sum_{u \in C_t} deg_{G_{a(t)}}(u)}{2e_{a(t)}}\right)^2$$

Hierarchical LFR Benchmark [Lancichinetti and Radicchi, 2008]

is a "realistic" random graphs model with a two level community structure.



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Parameters (relevant here) :

- $\mu = \mu_1 + \mu_2$: average proportion of edges from a micro-community...
- μ_1 : ... to another macro-community.
- μ_2 : ... to the same macro-community.

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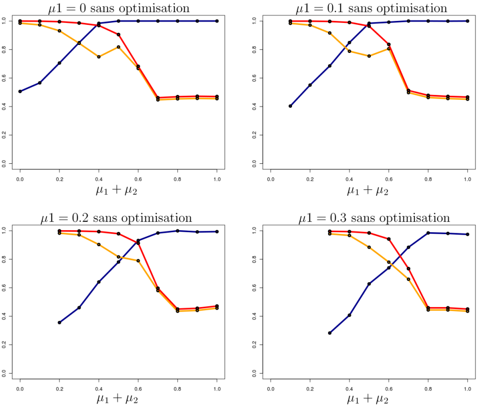
We want :

- $\mu_1, \mu_2 \simeq 0$: Only detect micro-communities.
- $\mu_1 < 0.5, \mu_2 \geq 0.5$: Only detect macro-communities.
- $\mu_1 < \mu_2 < 0.5$: Detect both micro and macro communities.

We compare our results to predictions using the *Variation of Information* distance:

- First produced level ($N_1(T)$) vs macro-communities
- Second produced level ($N_2(T)$) vs micro-communities
- Last produced level ($\mathcal{F}(T)$) vs micro-communities



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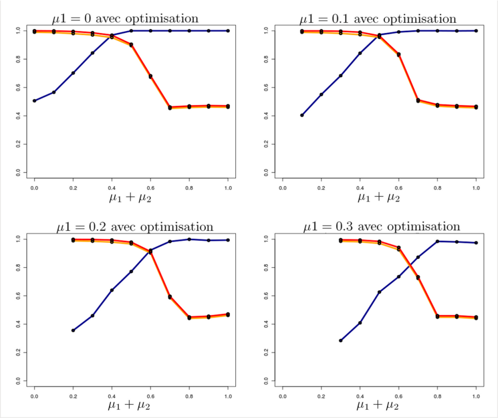
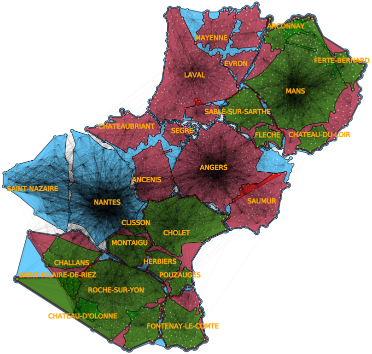
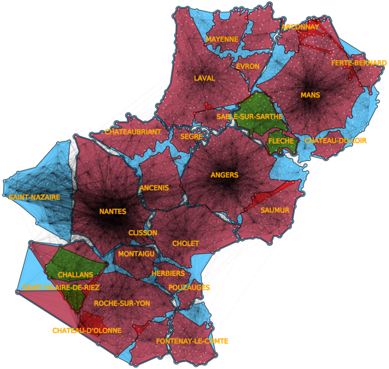


Pays-de-la-Loire commuters network (INSEE – 1999).





Conclusion: I introduced

- a **hierarchical quality measure** that naturally extend additives measures
- a procedure to **improve hierarchical partitions**

Limitations: The procedure

- is a greedy algorithm
- needs to be apply on different hierarchies (Tulip plugin available soon!)

Future directions:

- Find useful **properties** of $\Phi(G, T)$
- Define a **distance** metric between hierarchical partitions

Thanks for your attention.

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