Connectivity threshold of Bluetooth graphs

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Random geometric graphs

$G(n, r)$ $\qquad$ $N = \text{Poisson}(n)$ uniform points in $\mathcal{D} \subseteq \mathbb{R}^d$

$i \sim j$ iif $\|X_i - X_j\| < r$

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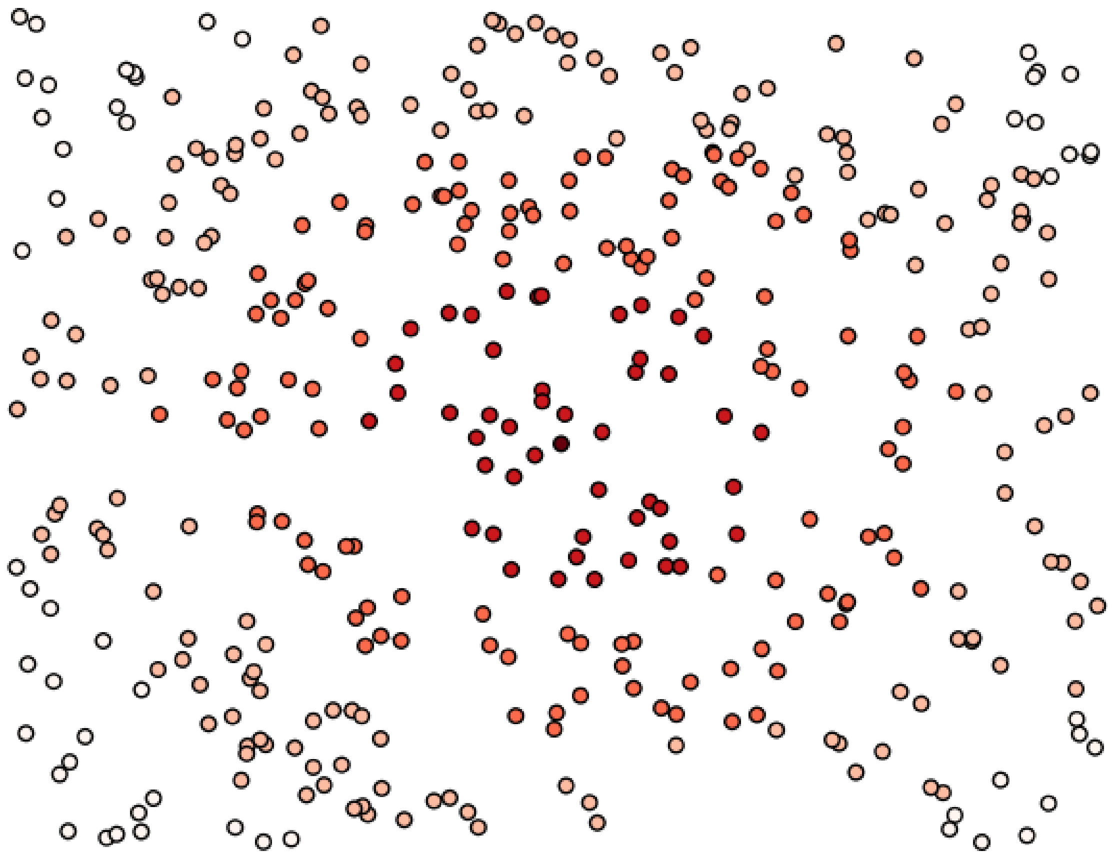

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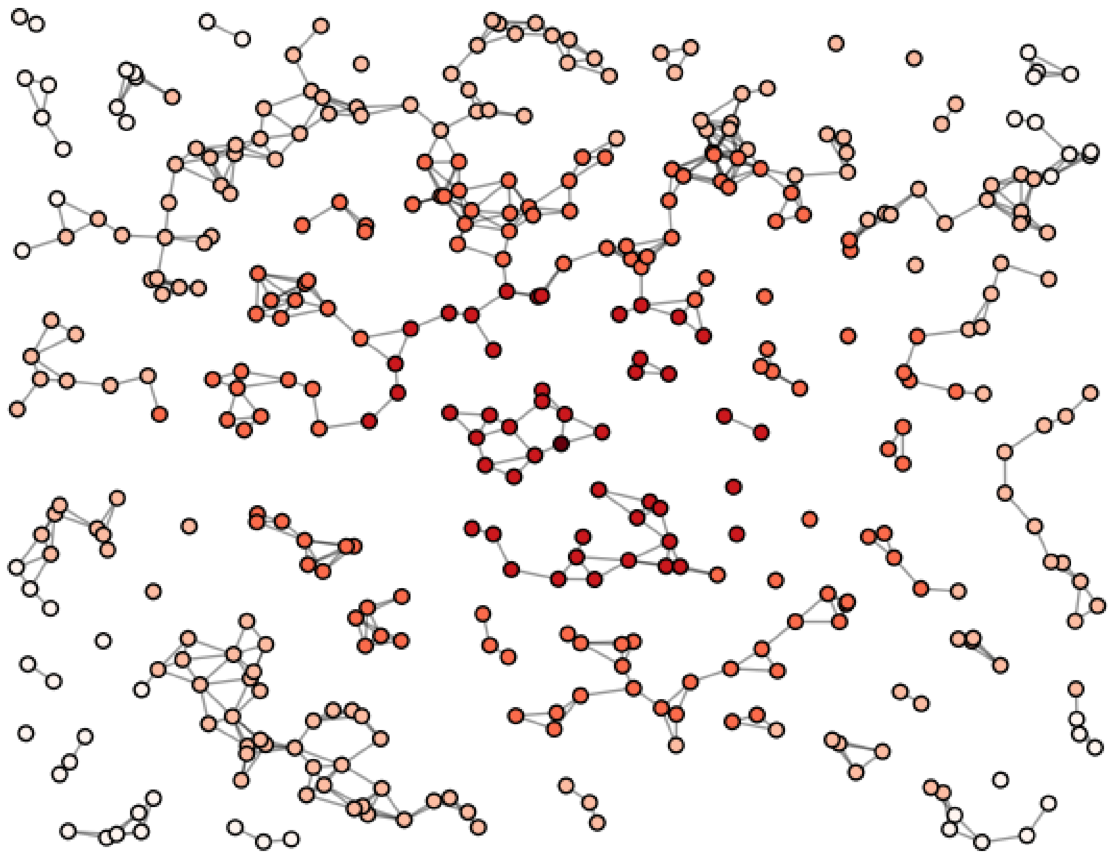

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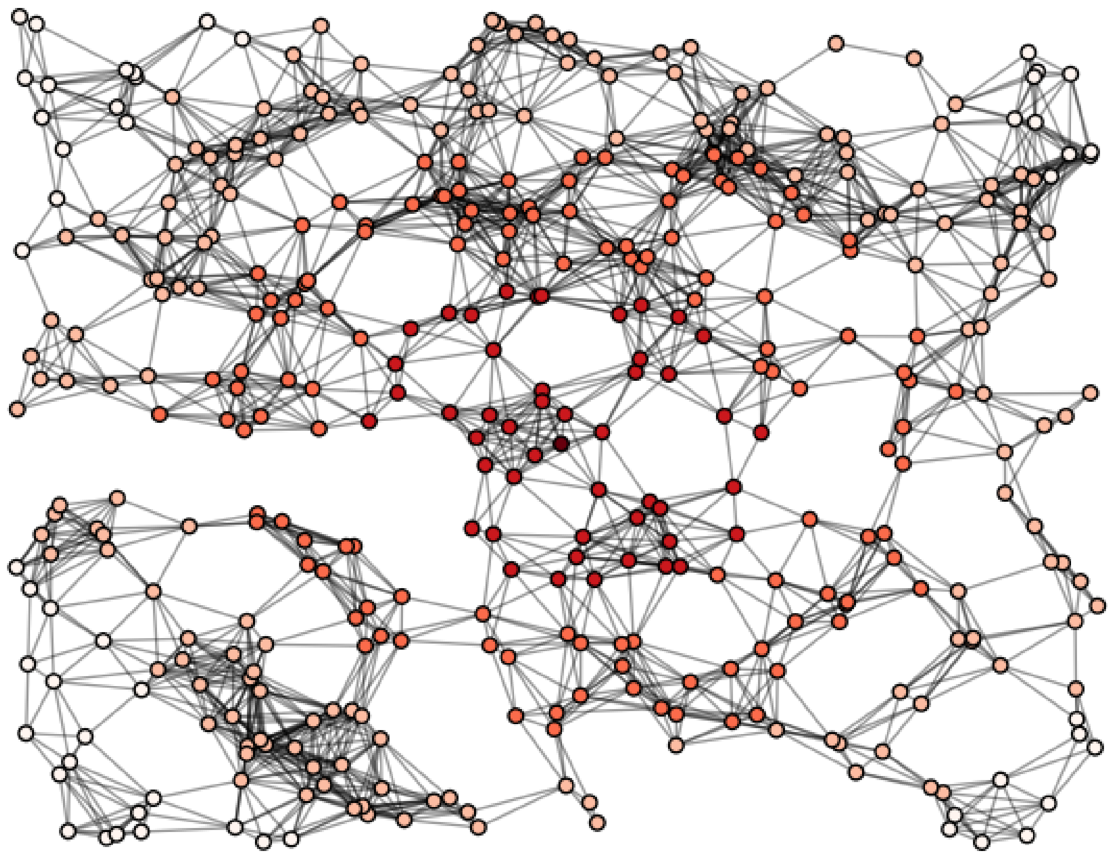

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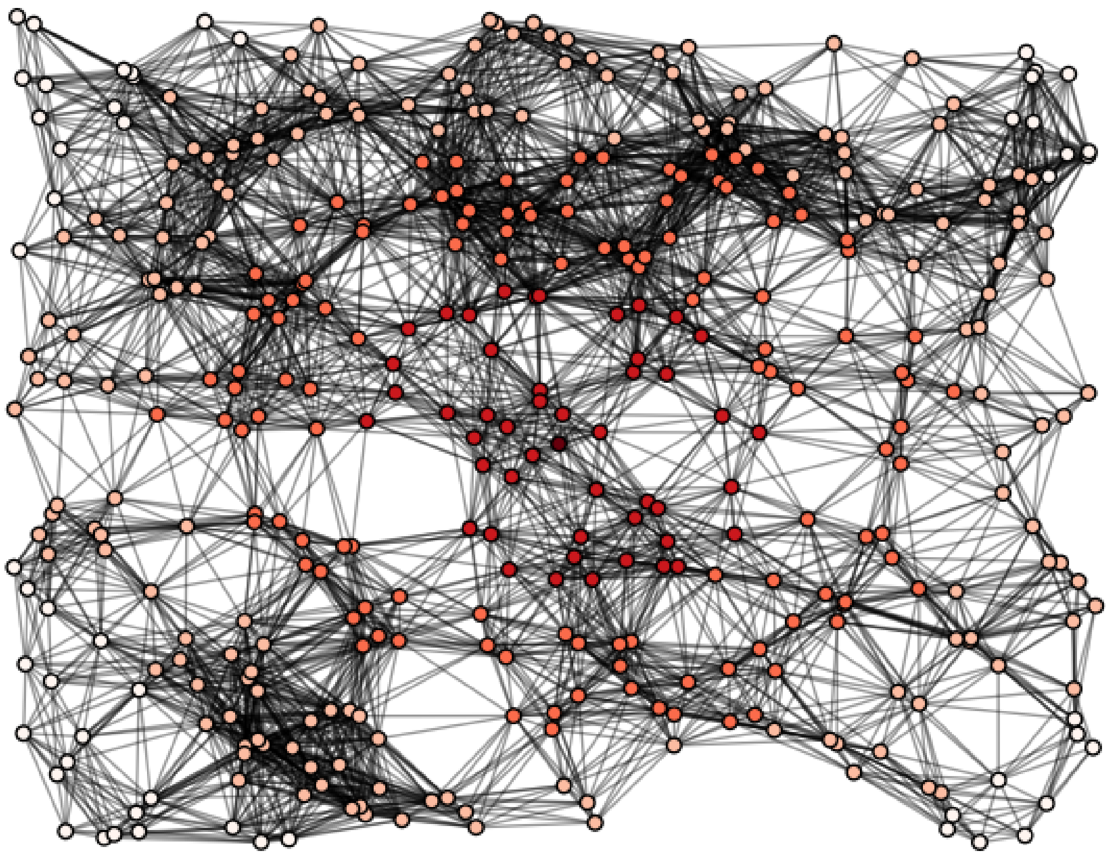

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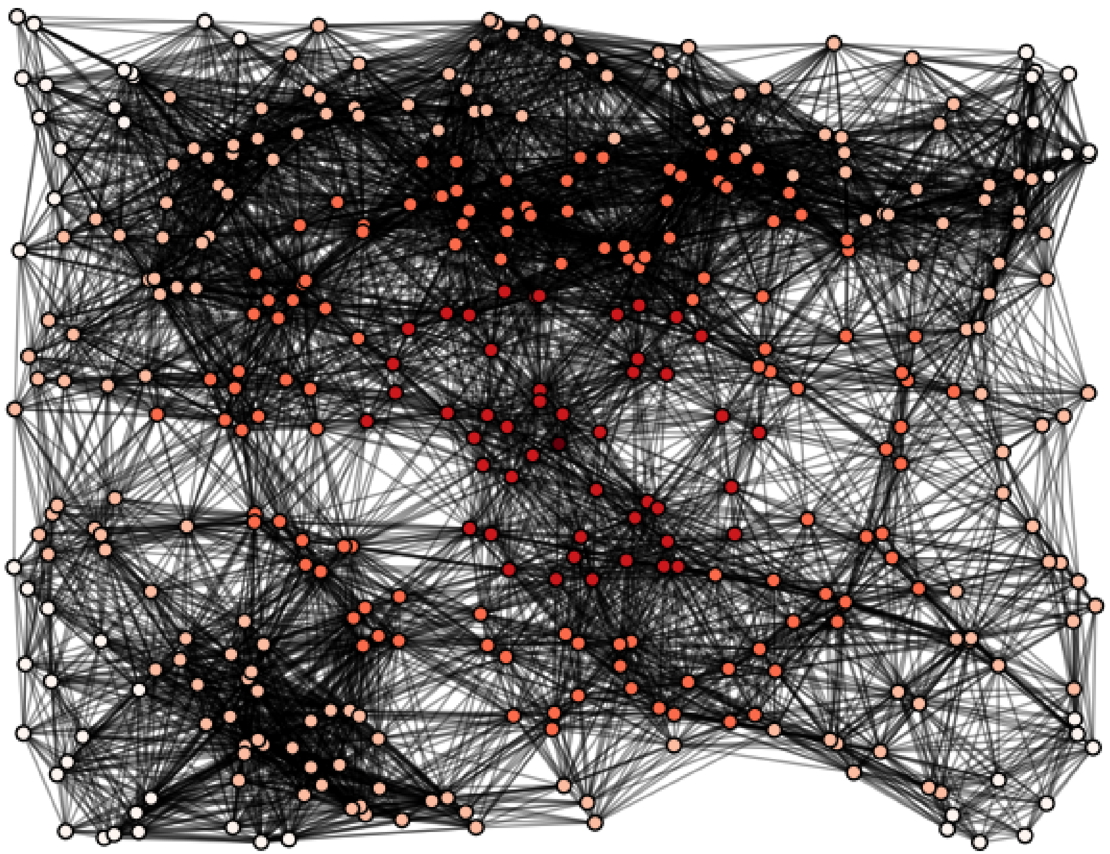

Random irrigation graphs

$G(n, r)$ gets connected when average degree is $\Theta(\log n)$

idea: sparsify the graph
in a distributed way
while ensuring connected

irrigation graphs $S_n(r, c)$

every point "sees" his neighbours in $G(n, r)$

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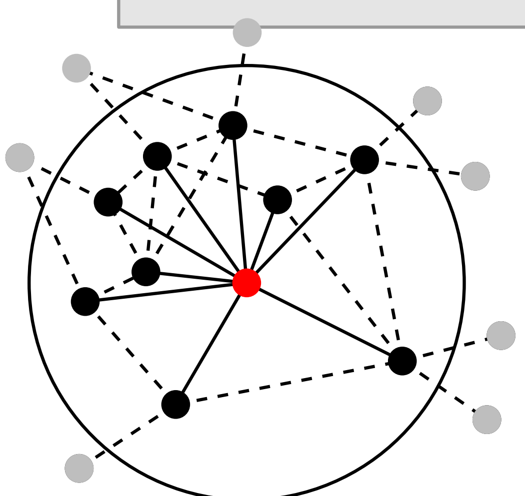

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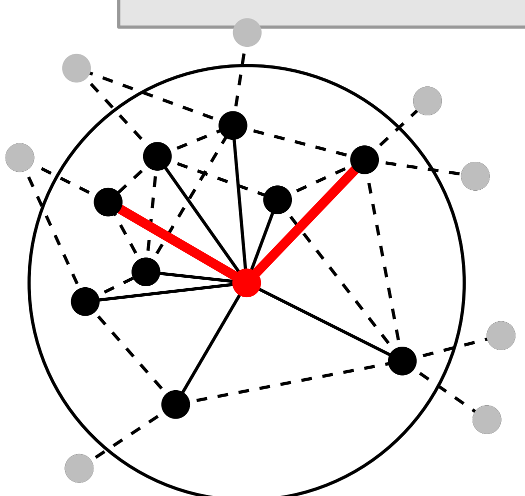

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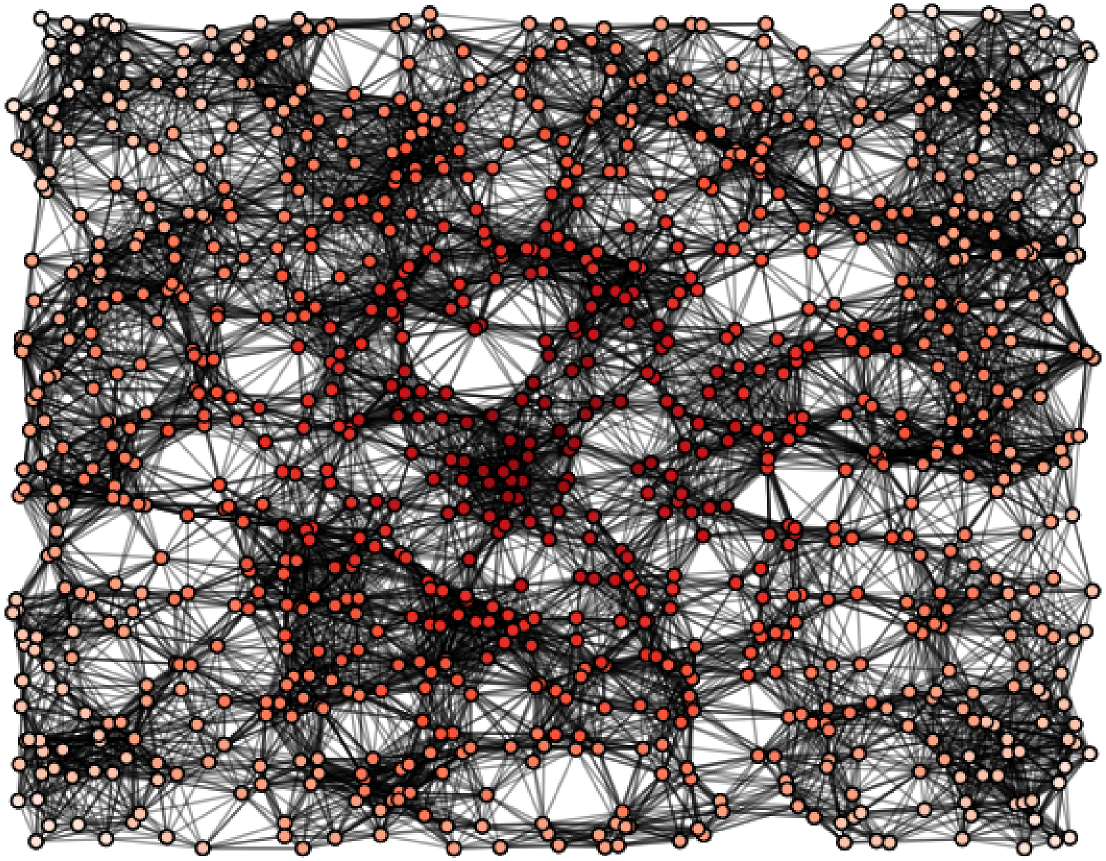

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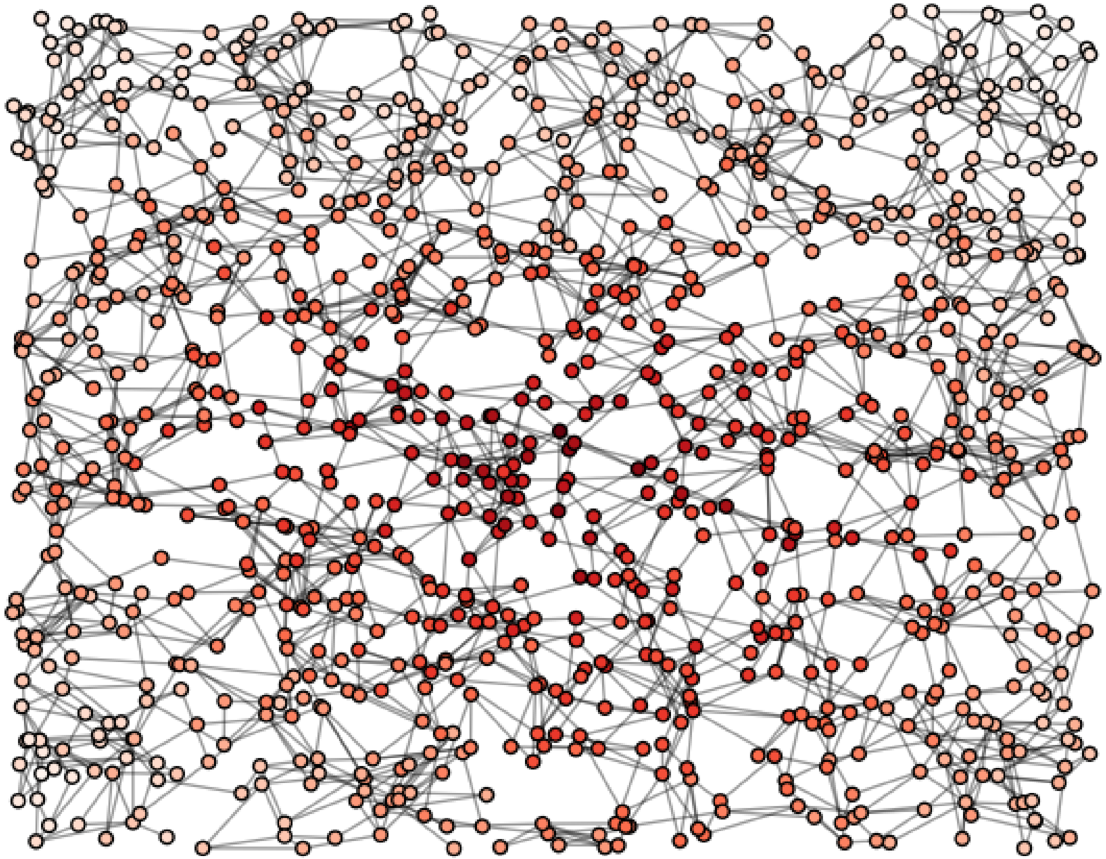


History and results

Previous results:

Dubhashi, Johansson, Häggström, Panconesi, and Sozio

$r = \Theta(1) \quad \Rightarrow \quad S_n(r, 2)$ is connected whp

BUT : expander! Fenner–Frieze

Crescenzi, Nocentini, Pietracaprina, Pucci

$d = 2 \quad r > \sqrt{\log n / n} \quad \Rightarrow \quad S_n(r, c)$ is connected whp

$c > \gamma_2 \log(1/r)$

Main result

Theorem. $\mathcal{D} = [0,1]^d \qquad \epsilon \in (0,1)$

$$r > \gamma \sqrt[d]{\log n / n} \qquad \frac{\log(nr^d)}{\log \log n} \to \lambda \in [1, \infty]$$

1. $$c = \left\lfloor \sqrt{(1-\epsilon)\left(\frac{\lambda}{\lambda - 1/2}\right)\frac{\log n}{\log(nr^d)}} \right\rfloor$$

Then $S_n(r,c)$ is disconnected whp

2. $$c = \left\lfloor \sqrt{(1+\epsilon)\left(\frac{\lambda}{\lambda - 1/2}\right)\frac{\log n}{\log(nr^d)}} \right\rfloor$$

Then $S_n(r,c)$ is connected whp

Connectivity of random graphs

Connectivity $\Leftrightarrow$ $\nexists$ a cut without edges

Typical model. $G(n, p)$ n vertices $\{1, 2, \ldots, n\}$

$i \sim j$ with proba p

Lower bound. find a cut that does not contain edges

Given that there is a cut, how large should it be?

number of cuts of size k $\binom{n}{k}$
probability that it is empty: $(1-p)^{k(n-k)}$

Strategy: Find the best possible "local" obstruction
gives a bound on the parameter p
Prove that above the graph is connected

Connectivity of random geometric graphs

Threshold for connectivity for $G(n, p)$

cheapest obstruction: isolated vertex

$$p \sim \frac{\log n}{n} \qquad \text{given by } n\mathbf{P}\left(\mathrm{Bin}(n-1, p) = 0\right) = 1$$

$$\Rightarrow \text{ average degree about } \log n$$

Threshold for connectivity of random geometric graphs:

empty cut $\approx$ tube of width r contaning no point

Cheapest obstruction: isolated vertex

$$r \sim \gamma\sqrt{\frac{\log n}{n}} \qquad \text{given by } n\mathbf{P}\left(B(x, r) = \varnothing\right) = 1$$

Lower bound: a cheap obstruction

Cheapest possible obstruction isolated $(c+1)$-clique

Fix $c+1$ vertices

$$\mathbf{P}\left(\text{isolated } (c+1)\text{-clique}\right) \geq \mathbf{P}\left(c+1 \text{ choose among them}\right)$$
$$\times \mathbf{P}\left(\text{no other chooses them}\right)$$

$$\mathbf{P}\left(c+1 \text{ choose among them}\right) \geq \left(\frac{c}{\beta \log n}\right)^{c(c+1)}$$

$$\mathbf{P}\left(\text{others don't pick them}\right) \geq \left(1 - \frac{c+1}{\alpha \log n}\right)^{\#}$$

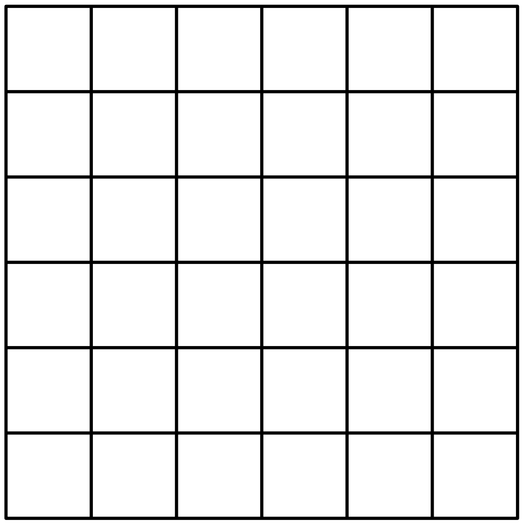
and $\# \leq c\beta \log n$

Find the value c such that $\dfrac{n}{c+1}\dbinom{\alpha \log n}{c} \times \mathbf{P}\left(\text{ok}\right) = 1$

Strategy for an upper bound

Need to "construct" the connectivity

1. discretize the square $[0,1]^2$ in Q_i, $1 \leq i \leq \lfloor 1/r \rfloor^2$
2. start from good local events
3. try to propagate connectivity to the entire graph



$$r \sim \gamma \sqrt{\frac{\log n}{n}}$$

$$\mathcal{P} = \{X_1, \dots, X_N\}$$

$$N \sim \mathrm{Poisson}(n)$$

Uniformity: whp, for all balls and cells

$$\alpha_1 n r^2 \leq \# \mathcal{P} \cap B(X_i, r) \leq \beta_1 n r^2$$

$$\alpha_1 n r^2 \leq \# \mathcal{P} \cap Q_i \leq \beta_1 n r^2$$

High level approach

Notation.

black if all points in it are connected w/o using the outside

*-connected: share at least a corner

connected: share a face and linked by an edge of S_n

Main strategy

Suppose:

1. all cells are occupied and connected to their neighbours
2. largest *-connected white component $\leq q$
3. smallest c.c. of S_n is at least s
4. every cell contains $\geq \lambda \log n$ points

Proposition. Suppose 1-4 hold with:

$$q = o\left(r^{-1/2}\right) \qquad \frac{s}{\lambda \log n} > q^2$$

Then, S_n is connected

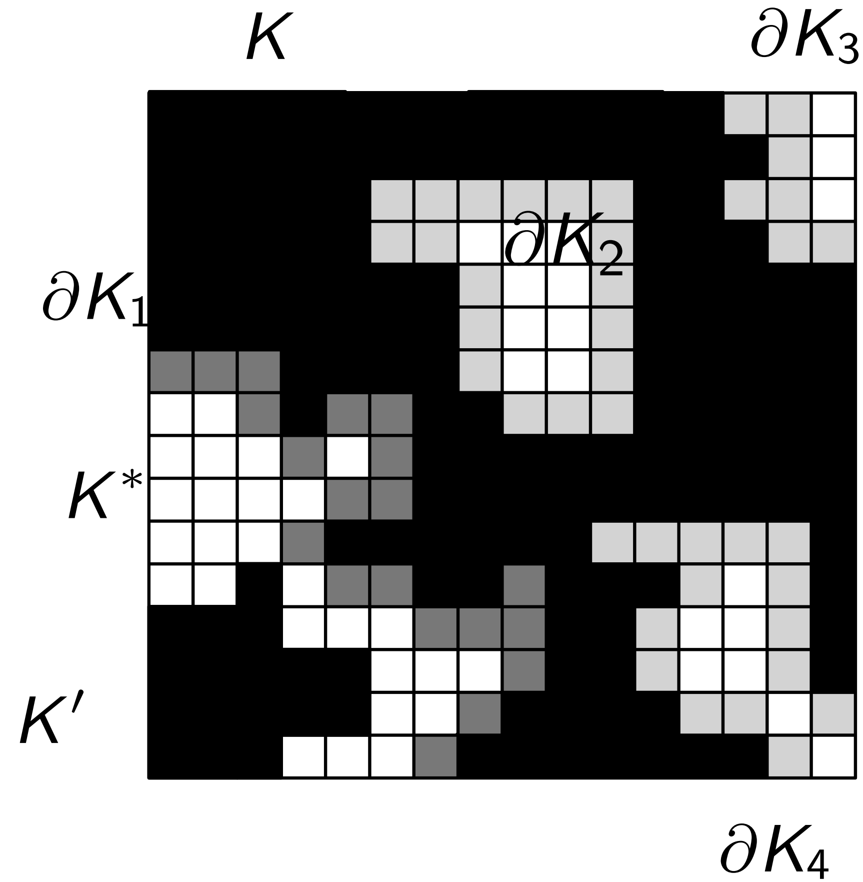
Sketch of proof

(a) exists a black crossing
of size $\geq 1/r$

Recolor blue the cells in small c.c.

(b) all remaining black cells
are connected

(isoperimetry)



(c) each vertex connects to the black c.c.

X_i not connected to the black c.c.
C the set of cells it touches
K^* the enlargement of C with white/blue cells
(isoperimetry)

The largest *-connected white component

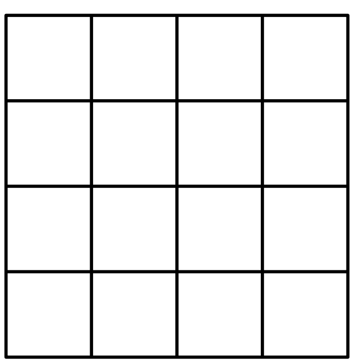
Aim. $q = 2(\log n)^{2/3}$ $\qquad$ $s = \exp((\log n)^{1/3})$

bound on the number of *-connected components of size k

$$\#\{\text{spanning trees of size } k\} \leq nC^k$$

if probability to be white $p \leq \exp(-(\log n)^{1/3})$

$$\mathbf{E}\left[\#\{\text{spanning trees } \geq q\}\right] \leq n \sum_{k \geq q} (pC)^k = O(1/n)$$



$(2d)^d$

push + pull

grow 2 neighborhoods of a single vertex
large enough for all other vertices to hook up

The smallest connected component of S_n

$$c = \sqrt{\frac{(2+\epsilon)\log n}{\log\log n}} \qquad \hat{c} = \sqrt{\frac{(2+\epsilon/2)\log n}{\log\log n}}$$

For $L = L(\epsilon)$ a constant, do L rounds:

1. round one each vertex selects $\hat{c}$ neighbors
2. in the following rounds, each vertex selects

idea: after round one, the smallest c.c. $\geq \delta \log n$

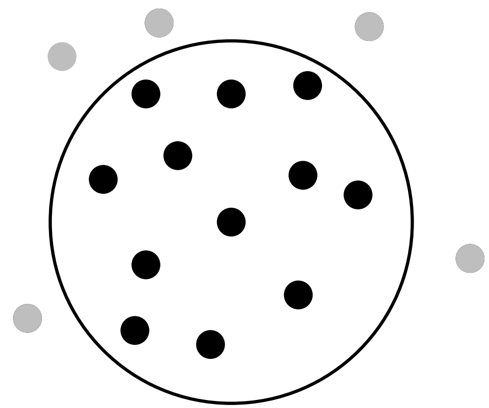
in the following rounds, each c.c. doubles

$$\frac{c-\hat{c}}{L} = \Delta\sqrt{\frac{2\log n}{\log\log n}} \qquad \delta \log n \times 2^{(c-\hat{c})/L} \geq \exp((\log n)^{1/3})$$

Smallest component of S_n – round 1

Bound expected number of c.c. of size h

$$\#\{\text{potential c.c. of size} h\} \leq n(\beta n r^2)^h$$



in a c.c. vertices must choose neighbors among themselves

Can choose δ independent of ϵ such that if $h \leq \delta \log n$ the probability that this occurs is too small

Smallest component of S_n – doubling rounds

Idea: small c.c. have a good chance to shoot outside

once it is large enough for this to fail: stop

by then, it is at least $n^{1/4} \geq \exp((\log n)^{1/3})$

How a c.c. C after round one populates the cells?

cell Q_i contains n_i points of C

m_i points of other components

Say a cell is red if $n_i > m_i$

1. not all cells can be red!
2. For neighbor cells red/blue $\mathbf{P}$ (no connection) $\leq n^{-L\xi}$
3. For neighbor blue cells $\mathbf{P}$ (no connection) $\leq n^{-L\xi}$

Finishing up

In **one** round

$$\mathbf{P}\left(\text{some c.c. does not connect}\right) \leq n^{1-L\xi}$$

In all the rounds

$$\mathbf{P}\left(\text{some round fails}\right) \leq \frac{c-\hat{c}}{L} \times n^{1-L\xi}$$

Choose the constant L large enough

And now...

Do we need full connectivity? ... super giant components

other models of “infection-like” graphs?

Bootstrap percolation on random geometric graphs

General approach based on branching arguments?

Thank you!