Connectivity threshold of Bluetooth graphs

Nicolas Broutin, *Inria Paris-Rocquencourt*

joint work with L. Devroye, *McGill*

N. Fraiman, *McGill*

G. Lugosi, *Pompeu Frabra*
Random geometric graphs

\[ G(n, r) \quad N = \text{Poisson}(n) \text{ uniform points in } \mathcal{D} \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ iif } \|X_i - X_j\| < r \]
Random geometric graphs

\[ G(n, r) \quad N = \text{Poisson}(n) \text{ uniform points in } D \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ iif } \|X_i - X_j\| < r \]
Random geometric graphs

\[ G(n, r) \]

\[ N = \text{Poisson}(n) \text{ uniform points in } D \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ iif } \|X_i - X_j\| < r \]
Random geometric graphs

\[ G(n, r) \]

\[ N = \text{Poisson}(n) \text{ uniform points in } \mathcal{D} \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ iif } \|X_i - X_j\| < r \]
Random geometric graphs

\[ G(n, r) \quad N = \text{Poisson}(n) \text{ uniform points in } \mathcal{D} \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ iiif } \|X_i - X_j\| < r \]
Random geometric graphs

\[ G(n, r) \quad N = \text{Poisson}(n) \text{ uniform points in } \mathcal{D} \subseteq \mathbb{R}^d \]

\[ i \sim j \text{ ii}f \quad \|X_i - X_j\| < r \]
Random irrigation graphs

\( G(n, r) \) gets connected when average degree is \( \Theta(\log n) \)

**idea:**
sparsify the graph
in a distributed way
while ensuring connected

**irrigation graphs** \( S_n(r, c) \)
every point ”sees” his neighbours in \( G(n, r) \)
every point keeps \( c \) neighbours chosen at random
Random irrigation graphs

$G(n, r)$ gets connected when average degree is $\Theta(\log n)$

**idea:** sparsify the graph in a distributed way while ensuring connected

**irrigation graphs** $S_n(r, c)$

- every point "sees" his neighbours in $G(n, r)$
- every point keeps $c$ neighbours chosen at random
Random irrigation graphs

\[ G(n, r) \] gets connected when average degree is \( \Theta(\log n) \)

**idea:** sparsify the graph in a distributed way while ensuring connected

**irrigation graphs** \( S_n(r, c) \)

- every point ”sees” his neighbours in \( G(n, r) \)
- every point keeps \( c \) neighbours chosen at random
Random irrigation graphs
Random irrigation graphs
History and results

Previous results:

Dubhashi, Johansson, Häggström, Panconesi, and Sozio

\[ r = \Theta(1) \implies S_n(r, 2) \text{ is connected whp} \]

BUT: expander! Fenner–Frieze

Crescenzi, Nocentini, Pietracaprina, Pucci

\[ d = 2 \quad r > \sqrt{\frac{\log n}{n}} \implies S_n(r, c) \text{ is connected whp} \]

\[ c > \gamma_2 \log(1/r) \]
Main result

Theorem. \( \mathcal{D} = [0, 1]^d \quad \epsilon \in (0, 1) \)

\[ r > \gamma d \sqrt{\log n / n} \quad \frac{\log(nr^d)}{\log \log n} \rightarrow \lambda \in [1, \infty] \]

1. \( c = \left\lfloor \sqrt{(1 - \epsilon) \left( \frac{\lambda}{\lambda - 1/2} \right) \frac{\log n}{\log(nr^d)}} \right\rfloor \)

Then \( S_n(r, c) \) is disconnected whp

2. \( c = \left\lfloor \sqrt{(1 + \epsilon) \left( \frac{\lambda}{\lambda - 1/2} \right) \frac{\log n}{\log(nr^d)}} \right\rfloor \)

Then \( S_n(r, c) \) is connected whp
Connectivity of random graphs

Connectivity  ⇔  ∄ a cut without edges

Typical model.  \( G(n, p) \)  \( n \) vertices \( \{1, 2, \ldots, n\} \)
\( i \sim j \) with proba \( p \)

Lower bound.  find a cut that does not contain edges

Given that there is a cut, how large should it be?

number of cuts of size \( k \) \( \binom{n}{k} \)
probability that it is empty: \( (1 - p)^{k(n-k)} \)

Strategy:  Find the best possible “local” obstruction
gives a bound on the parameter \( p \)
Prove that above the graph is connected
Connectivity of random geometric graphs

Threshold for connectivity for $G(n, p)$

cheapest obstruction: isolated vertex

$$p \sim \frac{\log n}{n} \quad \text{given by } n\mathbb{P}(\text{Bin}(n-1, p) = 0) = 1$$

$\Rightarrow$ average degree about $\log n$

Threshold for connectivity of random geometric graphs:

empty cut $\approx$ tube of width $r$ containing no point

Cheapest obstruction: isolated vertex

$$r \sim \gamma \sqrt{\frac{\log n}{n}} \quad \text{given by } n\mathbb{P}(B(x, r) = \emptyset) = 1$$
Lower bound: a cheap obstruction

Cheapest possible obstruction  isolated \((c + 1)\)-clique

Fix \(c + 1\) vertices

\[
\Pr(\text{isolated } (c + 1)\text{-clique}) \geq \Pr(c + 1 \text{ choose among them}) \\
\times \Pr(\text{no other chooses them})
\]

\[
\Pr(\text{choose among them}) \geq \left( \frac{c}{\beta \log n} \right)^{c(c+1)}
\]

\[
\Pr(\text{others don’t pick them}) \geq \left( 1 - \frac{c + 1}{\alpha \log n} \right) #
\]

and 
\[
# \leq c \beta \log n
\]

Find the value \(c\) such that 
\[
\frac{n}{c + 1} \left( \frac{\alpha \log n}{c} \right) \times \Pr(\text{ok}) = 1
\]
Strategy for an upper bound

Need to “construct” the connectivity

1. discretize the square $[0, 1]^2$ in $Q_i$, $1 \leq i \leq [1/r]^2$

2. start from good local events

3. try to propagate connectivity to the entire graph

\[ r \sim \gamma \sqrt{\frac{\log n}{n}} \quad \mathcal{P} = \{X_1, \ldots, X_N\} \]

\[ N \sim \text{Poisson}(n) \]

Uniformity: whp, for all balls and cells

\[ \alpha_1 nr^2 \leq \#\mathcal{P} \cap B(X_i, r) \leq \beta_1 nr^2 \]

\[ \alpha_1 nr^2 \leq \#\mathcal{P} \cap Q_i \leq \beta_1 nr^2 \]
High level approach

Notation.

black if all points in it are connected w/o using the outside
*-connected: share at least a corner
connected: share a face and linked by an edge of $S_n$
Main strategy

Suppose:

1. all cells are occupied and connected to their neighbours
2. largest *-connected white component $\leq q$
3. smallest c.c. of $S_n$ is at least $s$
4. every cell contains $\geq \lambda \log n$ points

**Proposition.** Suppose 1-4 hold with:

$$ q = o \left( r^{-1/2} \right) $$

Then, $S_n$ is connected

$$ \frac{s}{\lambda \log n} > q^2 $$
Sketch of proof

(a) exists a black crossing of size $\geq 1/r$

Recolor blue the cells in small c.c.

(b) all remaining black cells are connected

(isoperimetry)

(c) each vertex connects to the black c.c.

$X_i$ not connected to the black c.c.

$C$ the set of cells it touches

$K^*$ the enlargement of $C$ with white/blue cells

(isoperimetry)
The largest *-connected white component

**Aim.** \( q = 2(\log n)^{2/3} \quad s = \exp((\log n)^{1/3}) \)

bound on the number of *-connected components of size \( k \)

\[ \#\{\text{spanning trees of size } k\} \leq nC^k \]

if probability to be white \( p \leq \exp(-(\log n)^{1/3}) \)

\[ \mathbb{E} [\#\{\text{spanning trees } \geq q\}] \leq n \sum_{k \geq q} (pC)^k = O(1/n) \]

**push + pull**

grow 2 neighborhoods of a single vertex
large enough for all other vertices to hook up

\((2d)^d\)
The smallest connected component of $S_n$

\[
c = \sqrt{\frac{(2 + \epsilon) \log n}{\log \log n}}
\]

\[
\hat{c} = \sqrt{\frac{(2 + \epsilon/2) \log n}{\log \log n}}
\]

For $L = L(\epsilon)$ a constant, do $L$ rounds:

1. round one each vertex selects $\hat{c}$ neighbors

2. in the following rounds, each vertex selects

\[
\frac{c - \hat{c}}{L} = \Delta \sqrt{\frac{2 \log n}{\log \log n}}
\]

\[
\delta \log n \times 2^{(c - \hat{c})/L} \geq \exp((\log n)^{1/3})
\]

idea: after round one, the smallest c.c. $\geq \delta \log n$

in the following rounds, each c.c. doubles
Bound expected number of c.c. of size $h$

$$\#\{\text{potential c.c. of size } h\} \leq n(\beta nr^2)^h$$

in a c.c. vertices must choose neighbors among themselves

Can choose $\delta$ independent of $\epsilon$ such that if $h \leq \delta \log n$ the probability that this occurs is too small
Smallest component of $S_n$ – doubling rounds

**Idea:**
small c.c. have a good chance to shoot outside once it is large enough for this to fail: stop by then, it is at least $n^{1/4} \geq \exp((\log n)^{1/3})$

How a c.c. $C$ after round one populates the cells?

- cell $Q_i$ contains $n_i$ points of $C$
  - $m_i$ points of other components

Say a cell is red if $n_i > m_i$

1. not all cells can be red!
2. For neighbor cells red/blue $P$ (no connection) $\leq n^{-L\xi}$
3. For neighbor blue cells $P$ (no connection) $\leq n^{-L\xi}$
In one round

\[ P(\text{some c.c. does not connect}) \leq n^{1-L\xi} \]

In all the rounds

\[ P(\text{some round fails}) \leq \frac{c - \hat{c}}{L} \times n^{1-L\xi} \]

Choose the constant \( L \) large enough
And now...

Do we need full connectivity? ... super giant components other models of “infection-like” graphs?

Bootstrap percolation on random geometric graphs

General approach based on branching arguments?
Thank you!