Random Walks on Networks: Dynamics and Teleportation

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Generalized Master Equations for Non-Poissonian Dynamics on Networks, Till Hoffmann, Mason Porter and R.L.
Smart teleportation improves ranking and clustering of nodes in networks, R.L. and Martin Rosvall

naxys
Namur Center for Complex Systems

Networks

Metrics, statistics to describe their organization (degree distribution, modularity, etc.)

Models to reproduce the observed phenomena (PA) and build statistical ensembles of random networks (configuration model)

Effect of topology on dynamic processes, e.g. spreading of diseases.

Networks

Models for opinion formation, random walks, disease spreading, synchronization, etc.

Identification of “super-spreaders”, startegies to accelerate/hinder propagation, etc.

Effect of topology on dynamic processes, e.g. spreading of diseases.

Improve our understanding of how diffusion takes place in empirical systems

Networks

Metrics, statistics to describe their organization
(degree distribution, modularity, etc.)

Metrics based on
flows on the network

Models to reproduce the observed
phenomena (PA) and build
statistical ensembles of random
networks (configuration model)

Effect of topology on dynamic
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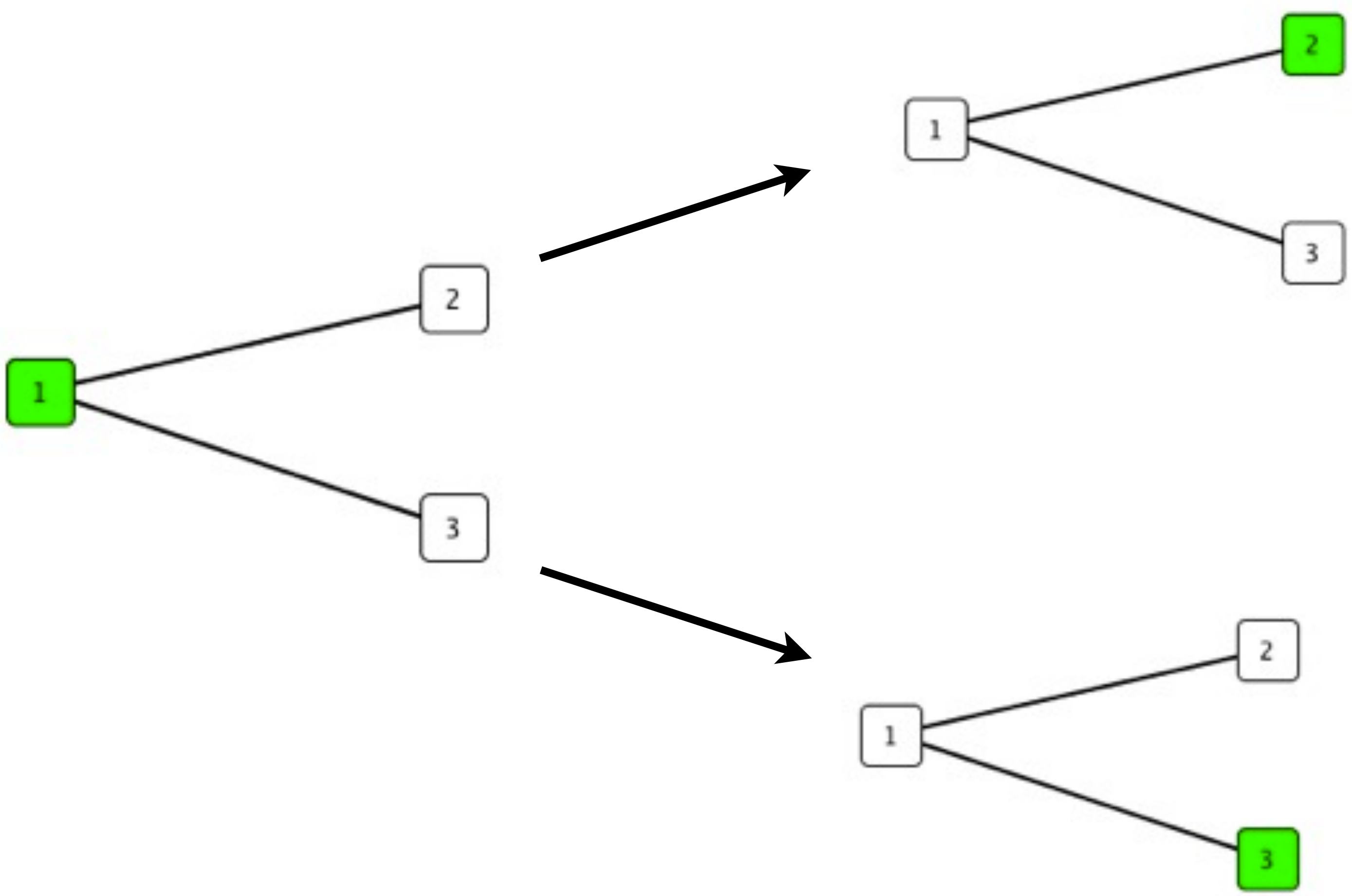
diseases.

Random Walk based metrics

Pagerank: the importance of a node is proportional to the density of random walkers on it at stationarity

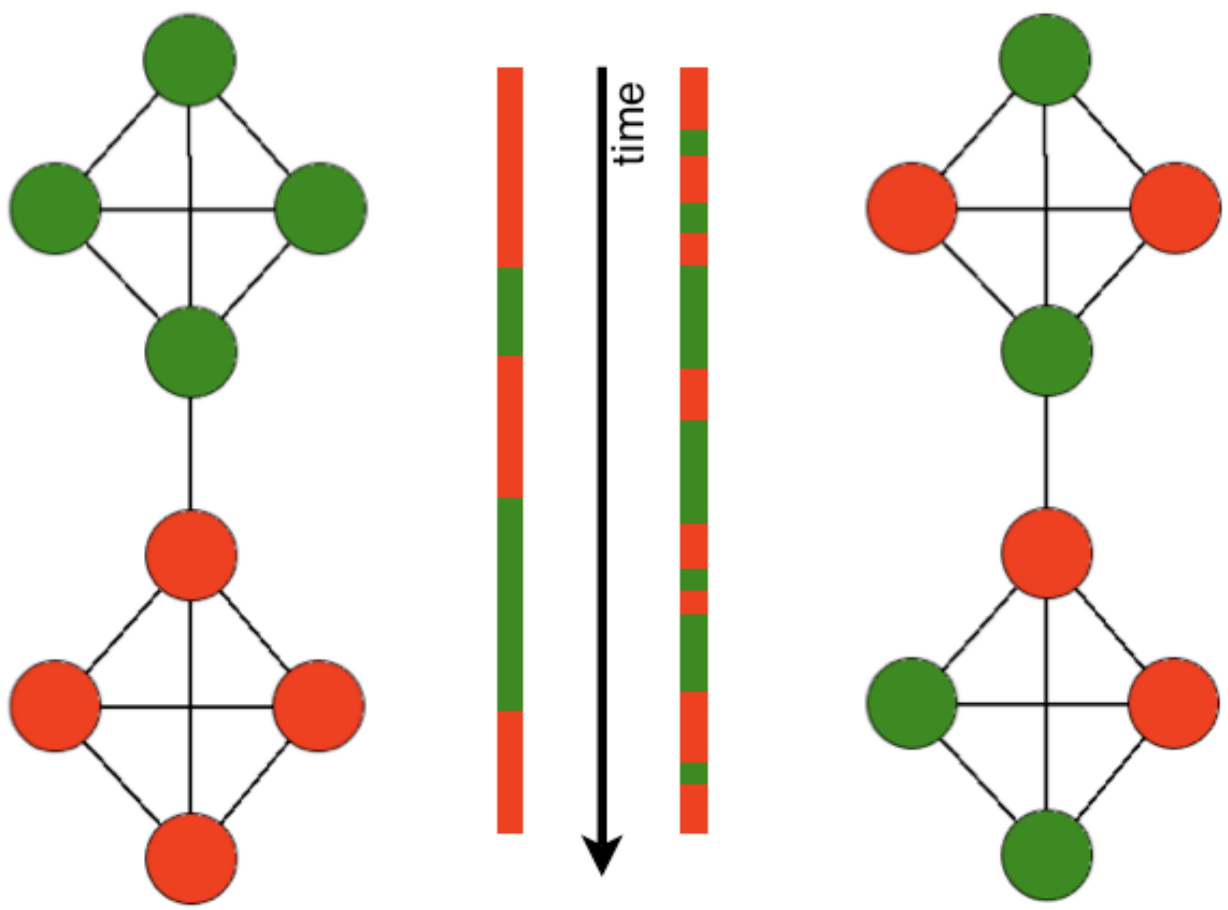
Random walkers explore the graph in an unbiased way



Random Walk based metrics

Map Equation and Stability

A flow of proability should be trapped for long time periods within a community before escaping it.



M. Rosvall and C. T. Bergstrom, PNAS 105, 1118 –1123 (2008)

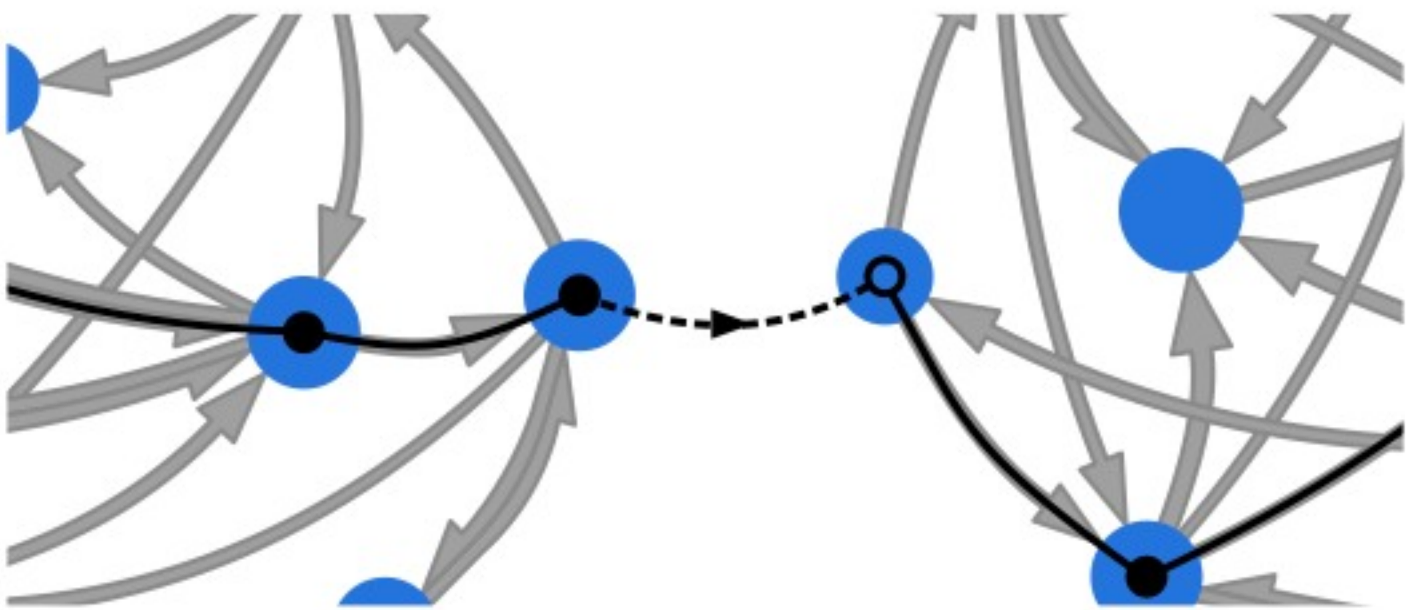
J.-C. Delvenne, S. Yaliraki & M. Barahona, Stability of graph communities across time scales. PNAS 2010

Random Walk based metrics are defined at stationarity

.... but the stationary state is either trivial, non-uniquely defined, or never reached in a majority of empirical systems

mathematical tricks have been proposed to make the dynamics ergodic

Occasional teleportation



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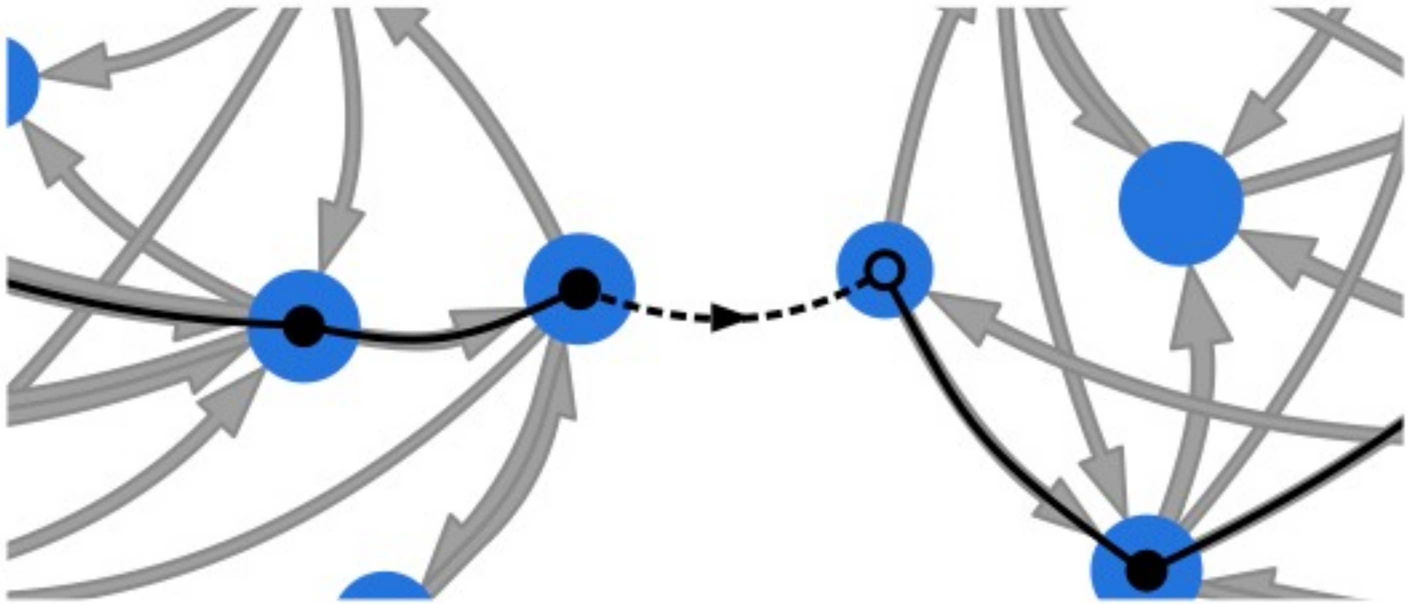
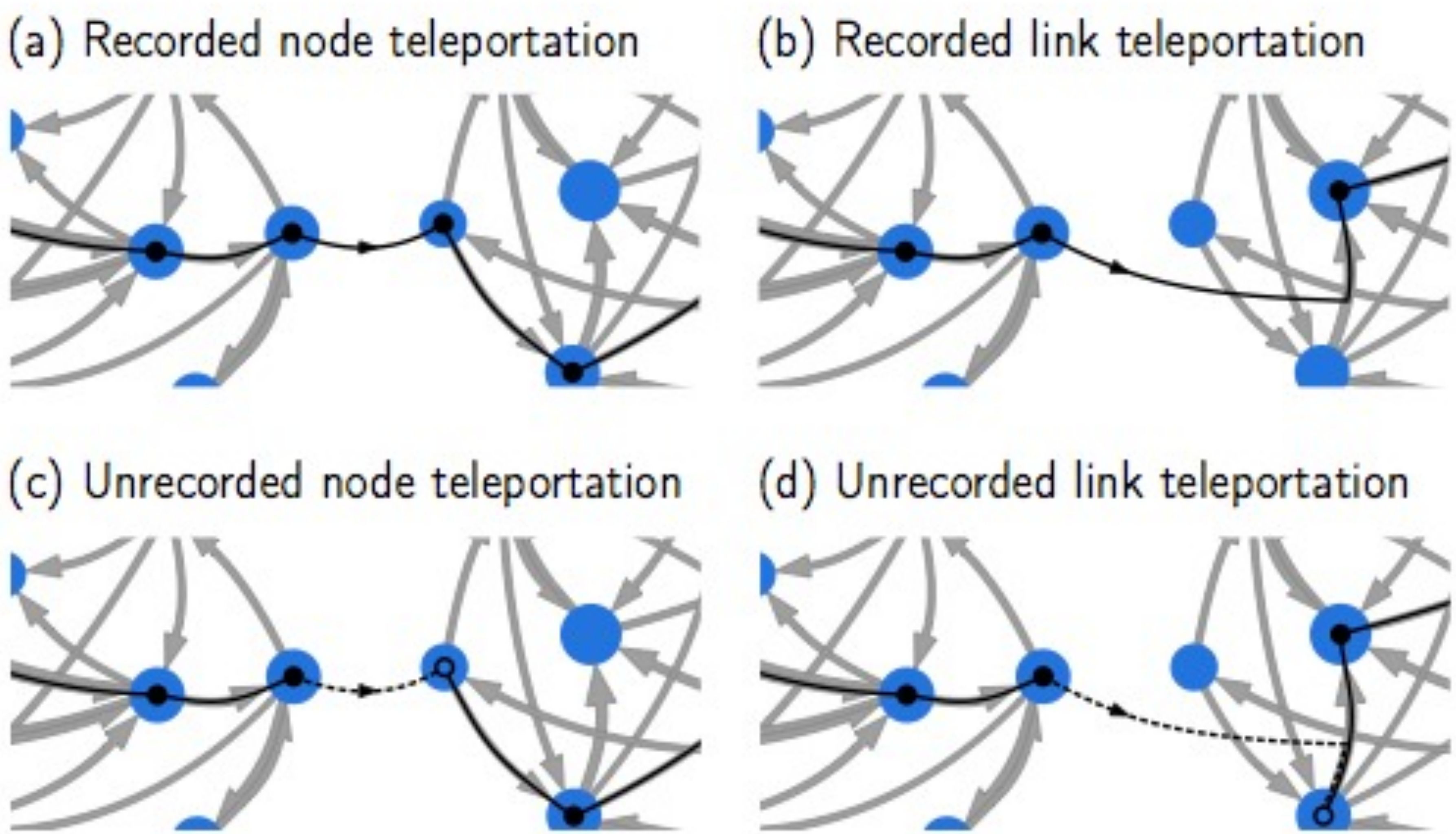


... but the ranking of nodes or their clustering into communities no longer only depends on the topological properties of the system, but also on the exact implementation of teleportation.

Smart teleportation

Is it possible to improve the standard teleportation scheme to minimize its effect on the outcome of the algorithm?

Non-uniform teleportation: teleportation depends on the topological properties of the nodes.



Pagerank

$$T_{ij} = A_{ij}/k_j^{\mathrm{out}}$$

Random walk without teleportation

$$p_{i;t+1} = \sum_j T_{ij} p_{j;t}.$$

Random walk with teleportation

$$p_{i;t+1} = \alpha \sum_j T_{ij} p_{j;t} + (1-\alpha) v_i,$$

$$\pi_{i;\alpha} = v_i + \sum_{k=1}^{\infty} \alpha^k \sum_j \left(T_{ij}^k - T_{ij}^{k-1}\right) v_j,$$

Pagerank depends on the details of teleportation, on vi and on the value of alpha

A vast majority of works tend to overlook these issues and use the standard value $\alpha = 0.85$ and the uniform preference vector $v_i = 1/N$, i.e. a walker randomly teleports on any node, independently on any intrinsic or topological properties. This choice of preference vec-

Smart teleportation

The preference vector depends on the topological properties of the graph:

$$v_i = \frac{k_i^{\mathrm{in}}}{L}$$

$$\pi_{i;\alpha} = \frac{k_i^{\mathrm{in}}}{L} + \sum_{k=1}^{\infty} \frac{\alpha^k}{L} \sum_j T_{ij}^k \left(k_j^{\mathrm{in}} - k_j^{\mathrm{out}}\right)$$

Robustness: pagerank is independent of alpha for

Undirected networks

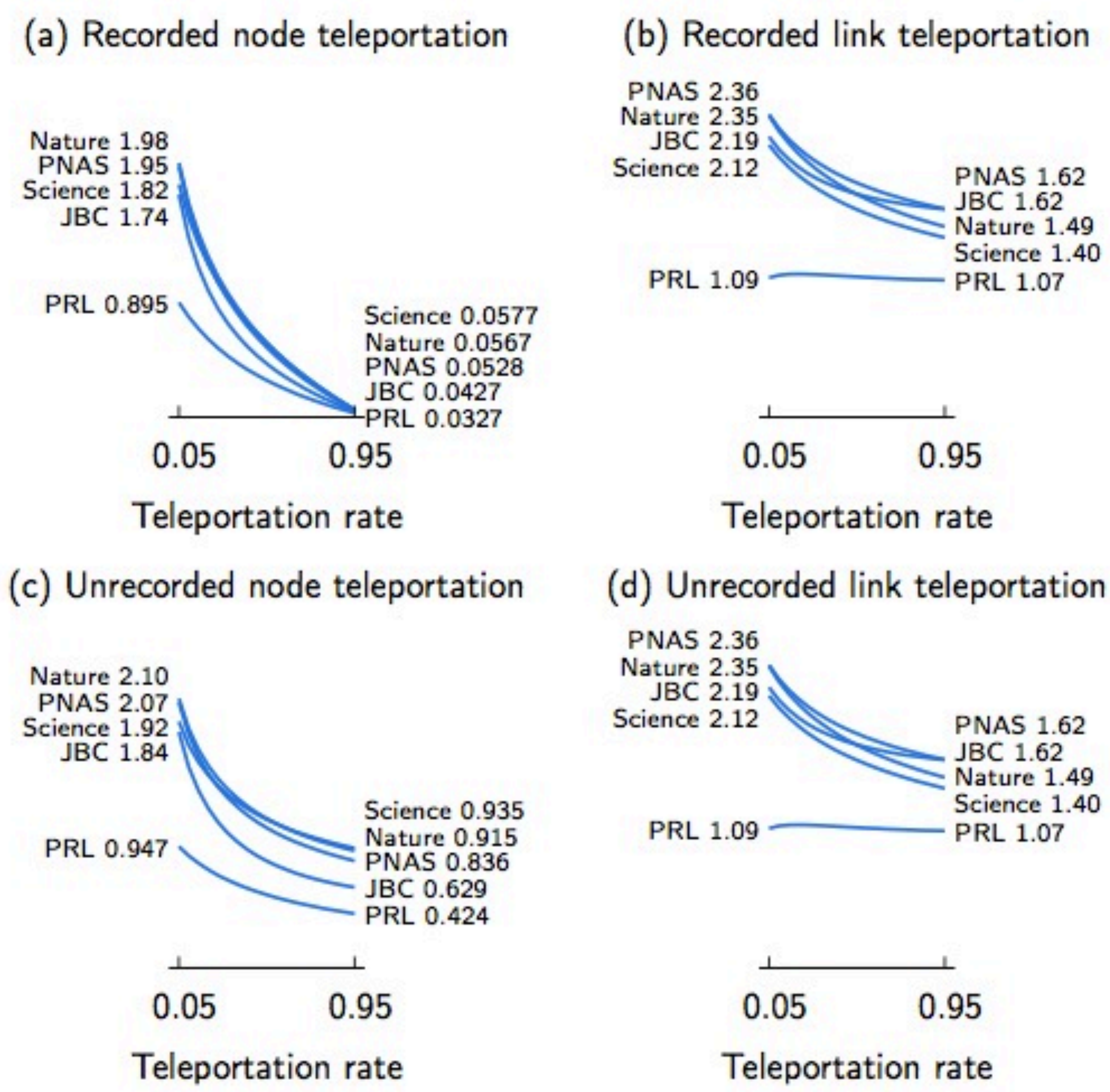
Eulerian graphs

Mean-field

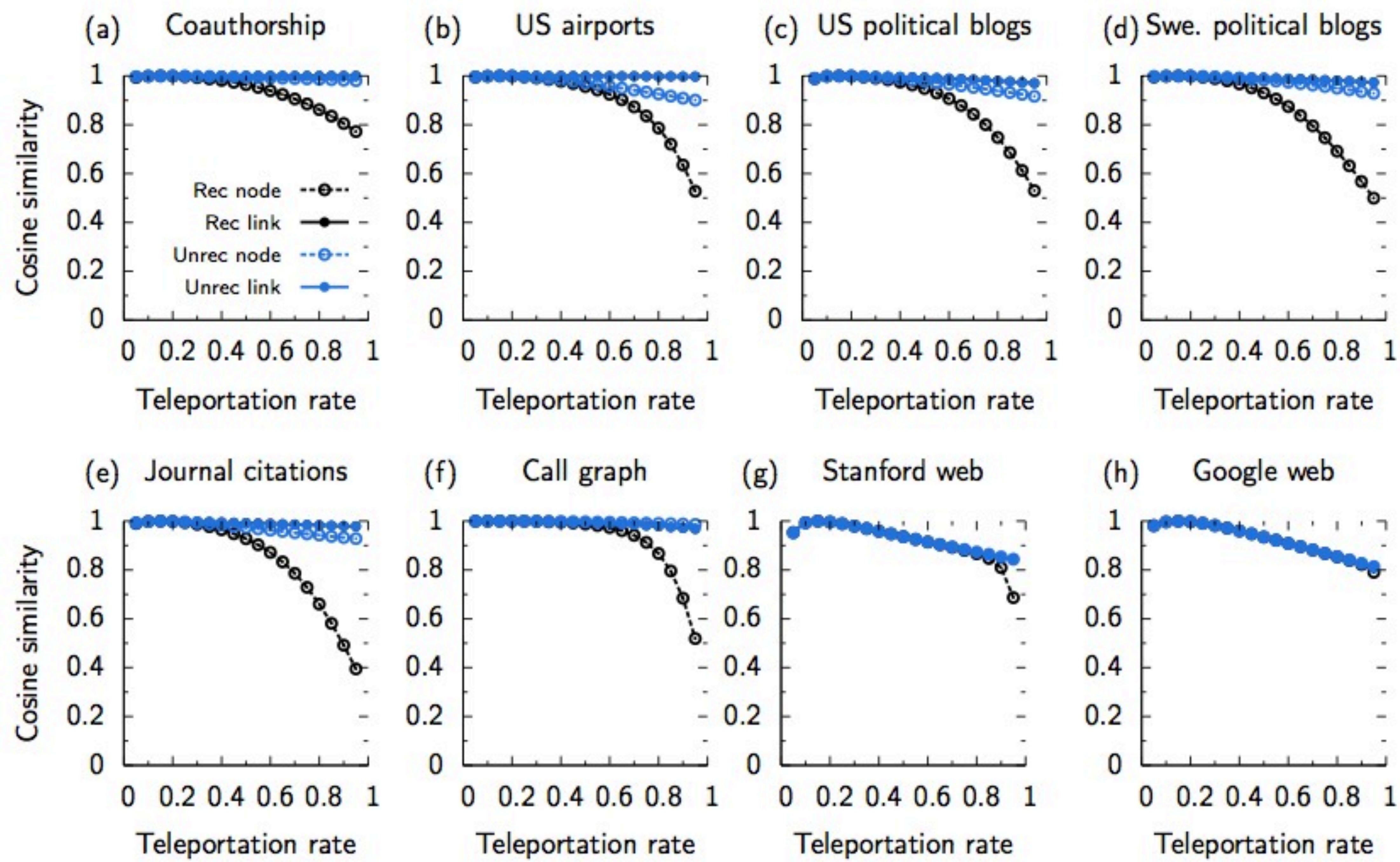
$$A_{ij} \approx k_i^{\mathrm{in}} k_j^{\mathrm{out}} / L$$

$$\boxed{\pi_i = k_i^{\mathrm{in}} / L}$$

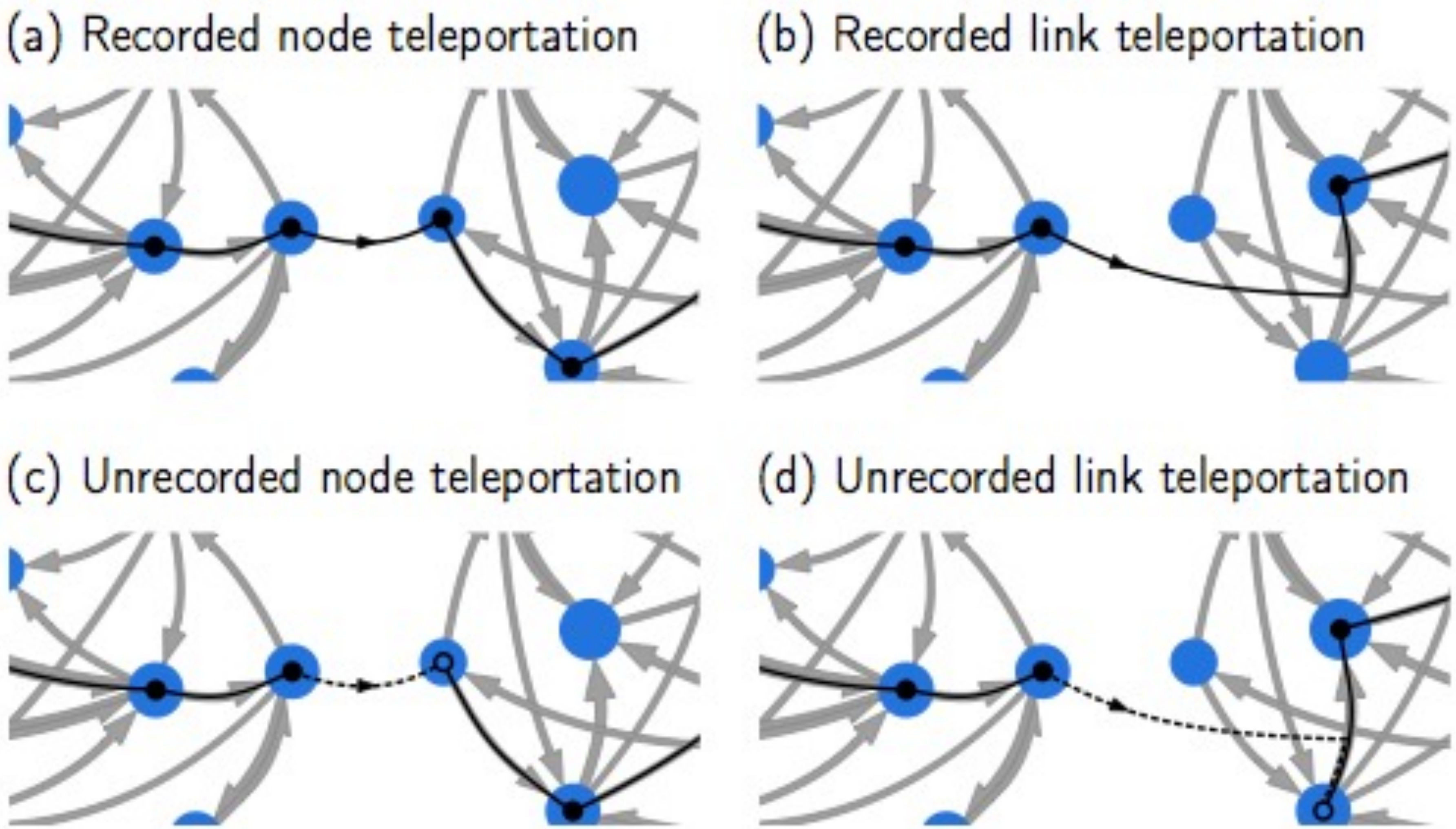
Empirical checks



Empirical checks



Community detection: don’t count teleportation steps



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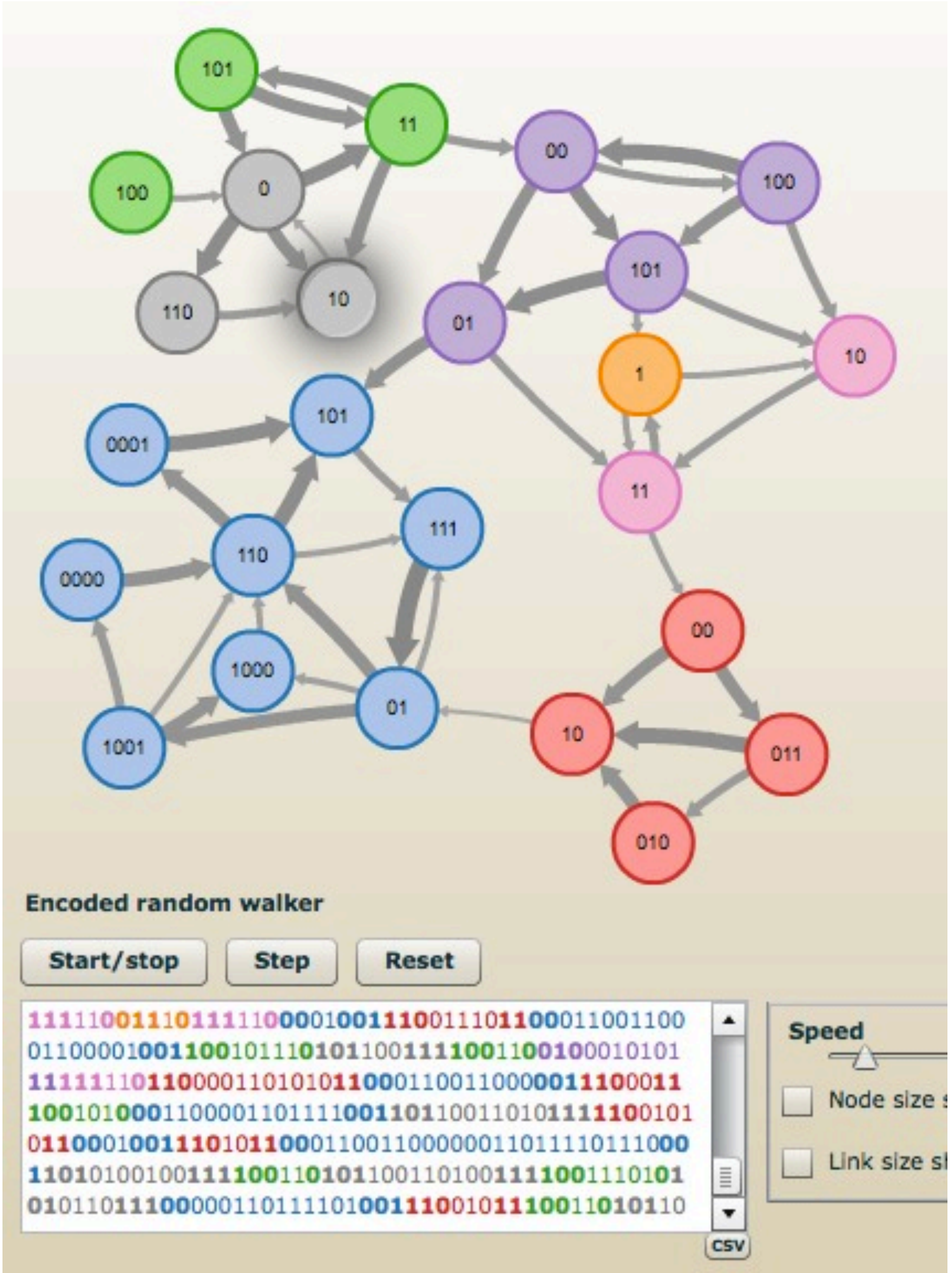

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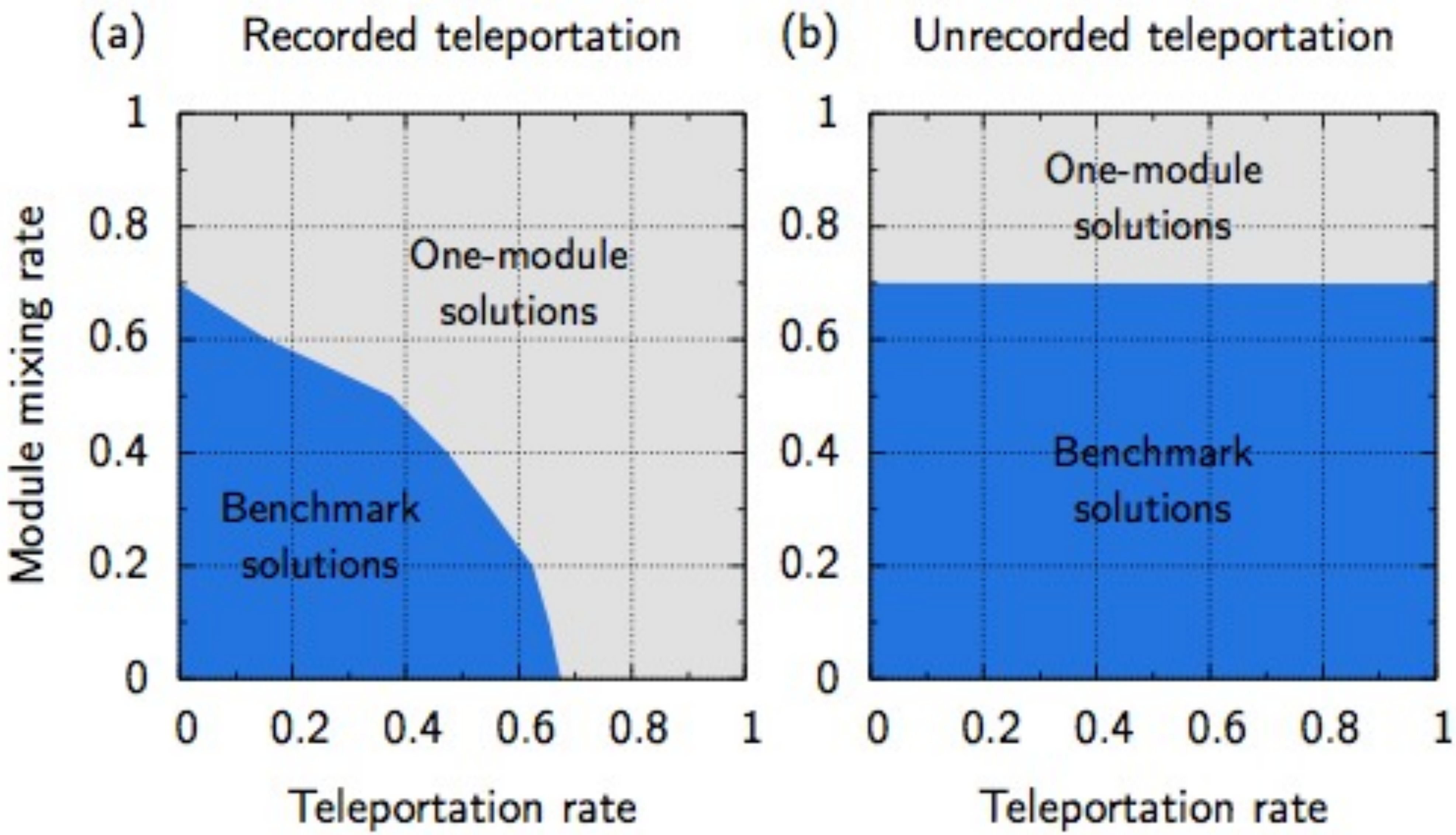
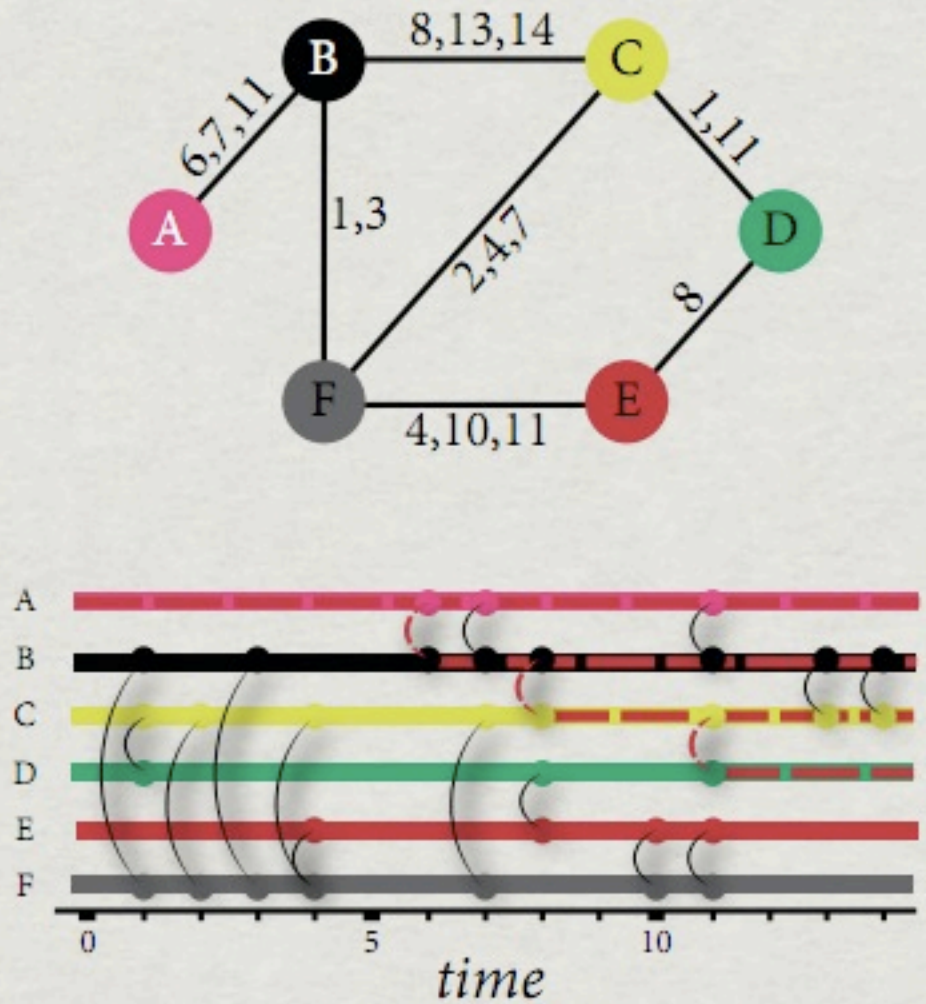


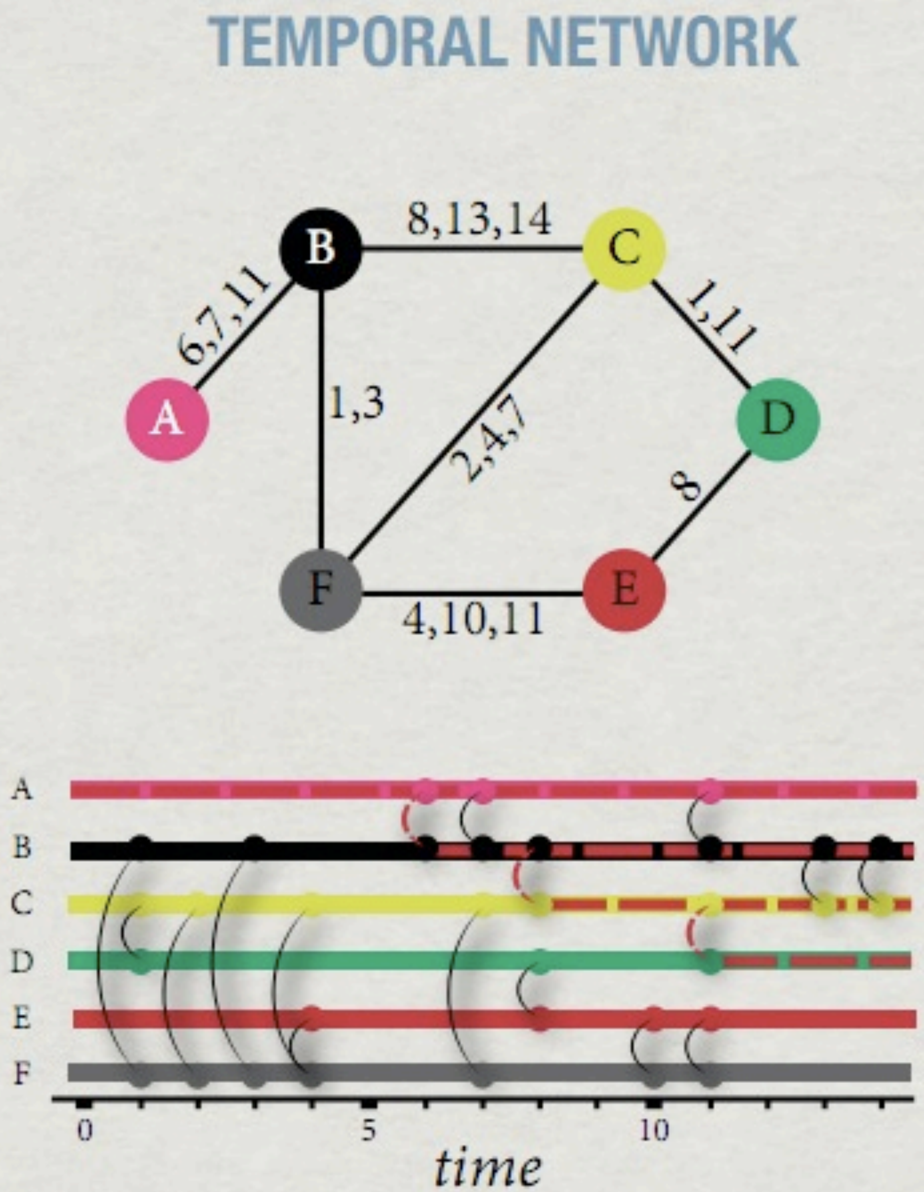
From static to temporal networks

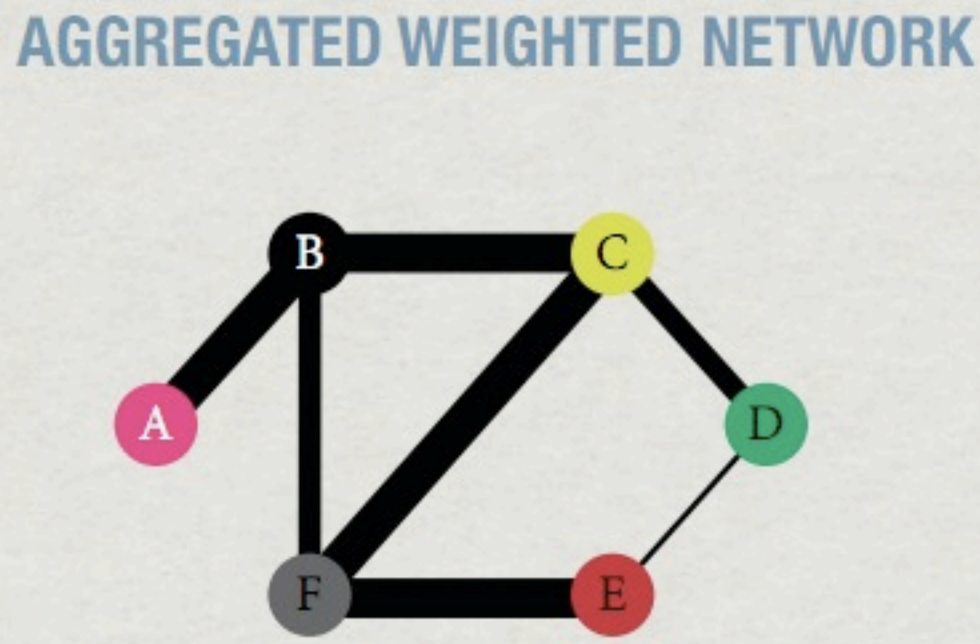
TEMPORAL NETWORK



see P. Holme & J. Saramäki, *Temporal Networks*, arXiv:1108.1780 (2011)

From static to temporal networks





see P. Holme & J. Saramäki, *Temporal Networks*, arXiv:1108.1780 (2011)

Dynamical networks: static viewpoint

Most works look at dynamics on aggregated networks, e.g. if a link appears 5 times between i and j in a certain time period, (i,j) receives a weight 5

Dynamics either occurs at discrete times

$$p_{i;n+1} = \sum_j \frac{A_{ij}}{k_j} p_{j;n}$$

or at continuous times as a Poisson process, with a rate proportional to the weight

$$\dot{p}_i = \sum_j \frac{A_{ij}}{k_j} p_j - p_i$$

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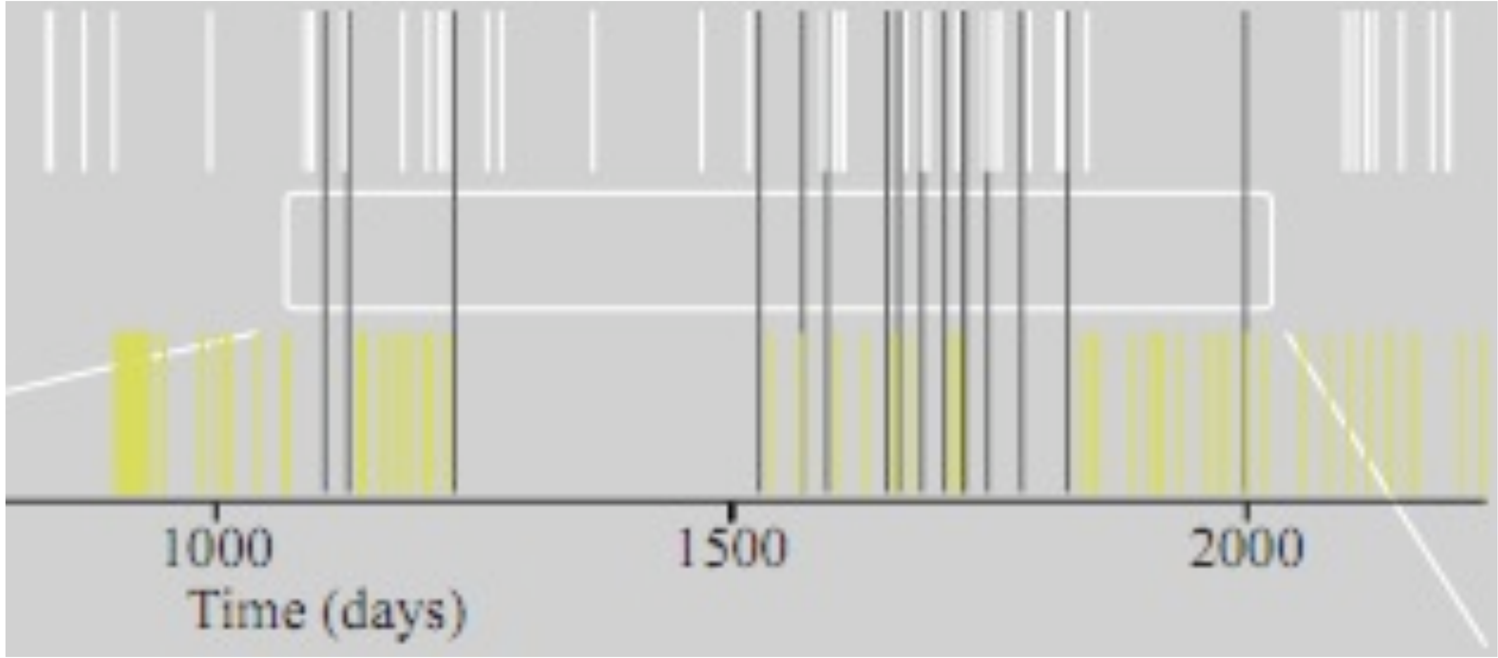
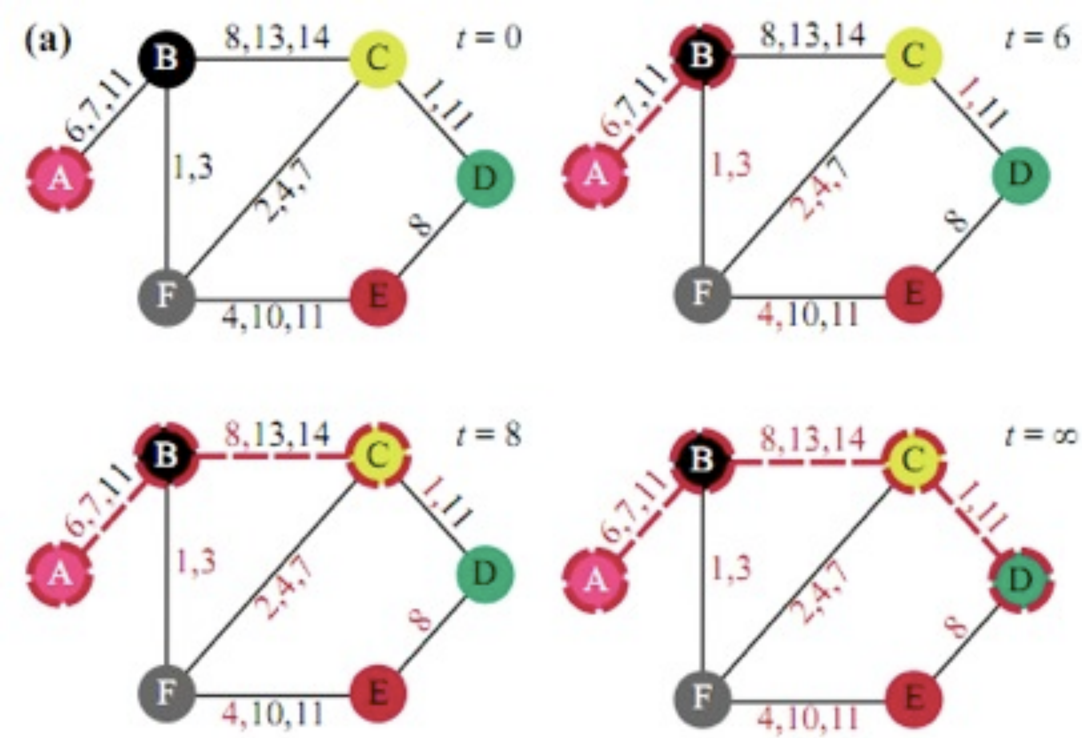


Simulations on temporal graphs...

Simulations on temporal graphs



Temporal Networks, P. Holme and J. Saramäki, arXiv: 1108.1780

... and comparison with null models

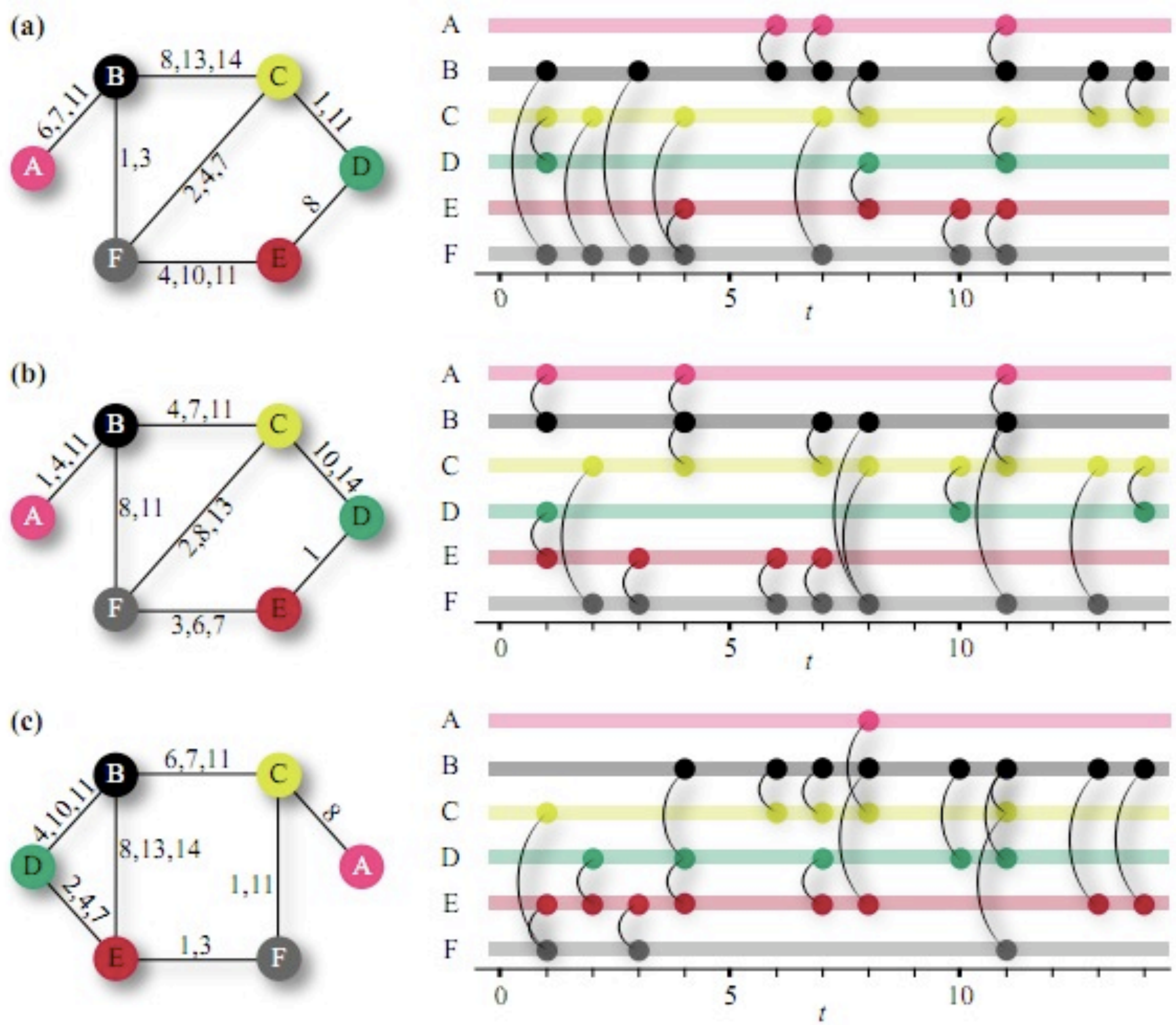


FIG. 11: Illustration of two types of randomization null-models for contact sequences. (a) shows a contact sequence (the same as in Fig. 1). In (b) it is randomized by the Randomly Permuted times procedure such that contacts happen the same number of time per edge, and the aggregated network topology is the same. In (c) the contact sequence in (a) is randomized by the Randomized edges (RE) procedure. With RE, the time sequence of the contacts along an edge is conserved, and so is the degree sequence of the original network, but all other structure of the topology is destroyed. (The latter statement is perhaps not so well illustrated by this figure as there are not so many graphs with the degree sequence of the original, aggregate graph.)

... and comparison with null models

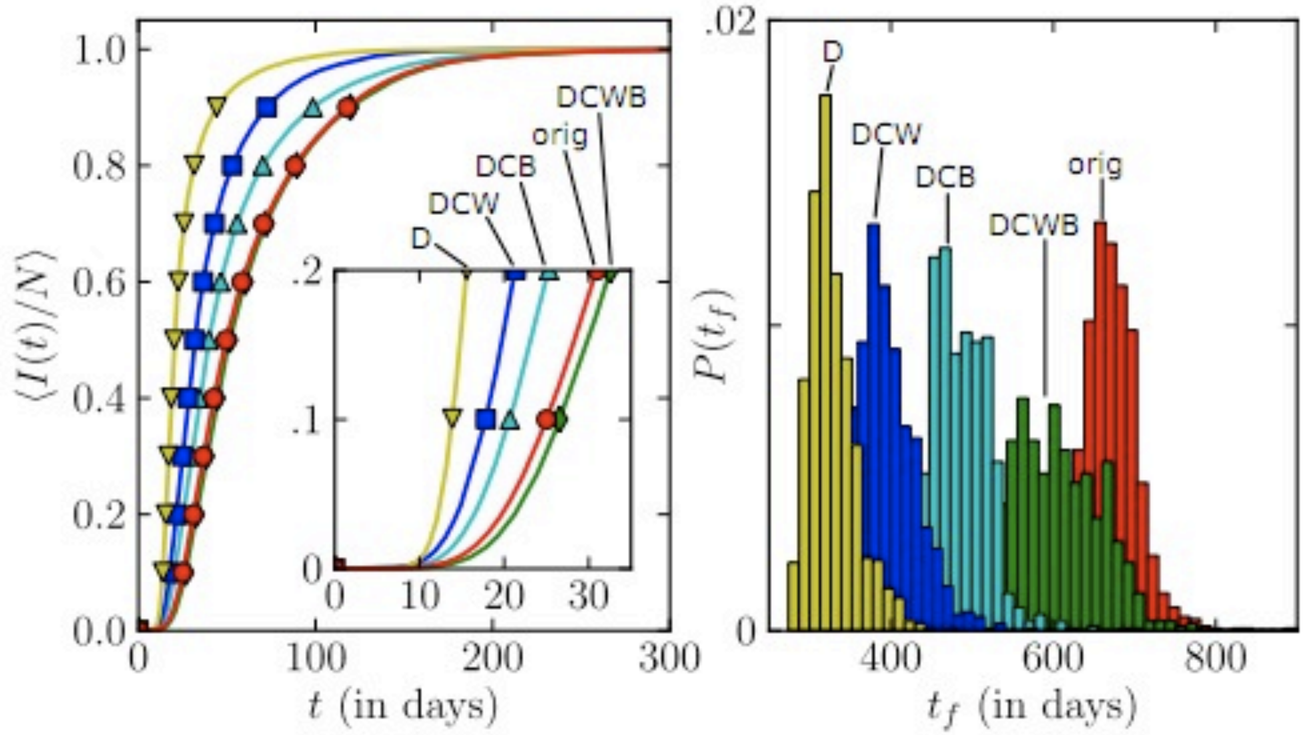


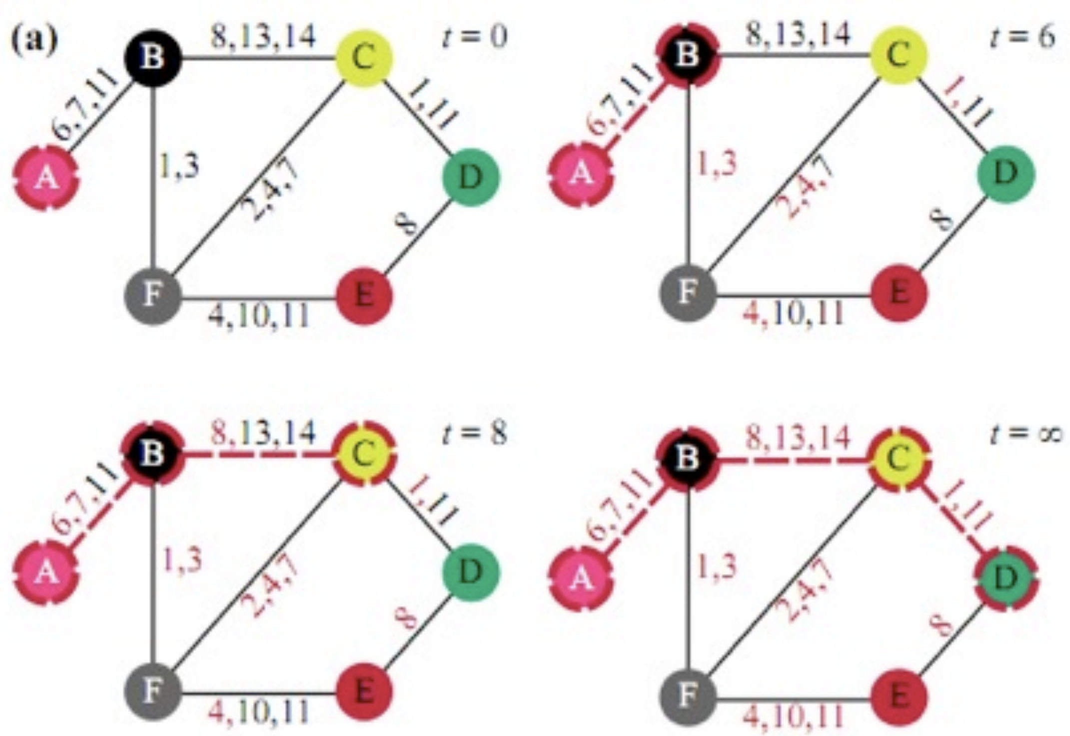
FIG. 1: (color online) (Left) Fraction of infected nodes $\langle I(t)/N \rangle$ as a function of time for the original event sequence ($\circ$) and null models: equal-weight link-sequence shuffled DCWB ($\Diamond$), link-sequence shuffled DCB ($\triangle$), time-shuffled DCW ($\square$) and configuration model D ($\triangledown$). Inset: $\langle I(t)/N \rangle$ for the early stages, illustrating differences in the times to reach $\langle I(t)/N \rangle = 20\%$. (Right) Distribution of full prevalence times $P(t_f)$ due to randomness in initial conditions.

... but lack of theoretical understanding

- slowing down compared to random times:
  see e.g. *Small but slow world: how network topology and burstiness slow down spreading*, M. Karsai et al, Phys Rev E 83, 025102(R) (2011); *Dynamical strength of social ties in information spreading*, Miritello et al, Phys. Rev. E 83, 045102(R) (2011)

- faster than random reference: see
  Simulated Epidemics in an Empirical Spatiotemporal Network of 50,185 Sexual Contacts, L.E.C. Rocha et al, PLoS Comput. Biol. 7, e1001109 (2011)

Dynamics on stochastic networks



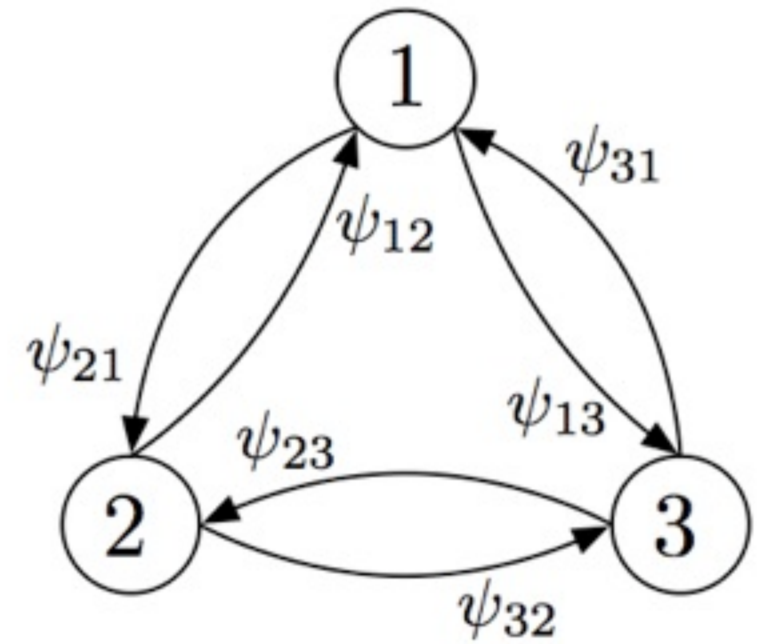


FIG. 1: A directed graph with $N = 3$ nodes and no self-loops. The waiting-time distribution $\psi_{ij}(t)$ characterizes the appearances of an edge from j to i.

Temporal Networks, P. Holme and J. Saramäki, arXiv: 1108.1780

Replacing the sequence of activation times by inter-event distributions
Deterministic => stochastic
Advantage: allows for a mathematical analysis and more realistic that simple weights

Random walks on stochastic networks

Generalized Montroll-Weiss Equation (usually for CTRW with non-Poisson inter-event time statistics on lattices)

$$\hat{n}(s) = \frac{1}{s}\left(I - \hat{D}_T(s)\right)\left(I - \hat{T}(s)\right)^{-1} n(0)$$

$$\frac{dn}{dt} = \left(T(t) * \mathcal{L}^{-1}\left\{\hat{D}_T^{-1}(s)\right\} - \delta(t)\right) * K(t) * n(t)$$

Convolution in time

Memory kernel

Effective transition matrix

Generalized Master Equations for Non-Poissonian Dynamics on Networks, T.H., M.A.P. and R.L.

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Convolution in time

Memory kernel

Effective transition matrix

$$\boxed{\begin{aligned} T_{ij}(t) &= \psi_{ij}(t) \times \prod_{k \neq i} \chi_{kj}(t) \\ &= \psi_{ij}(t) \times \prod_{k \neq i}\left(1 - \int_0^t \psi_{kj}(t')\, dt'\right). \end{aligned}}$$

Time ordering

Generalized Master Equations for Non-Poissonian Dynamics on Networks, T.H., M.A.P. and R.L.

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$$\Big\downarrow \text{Poisson}$$

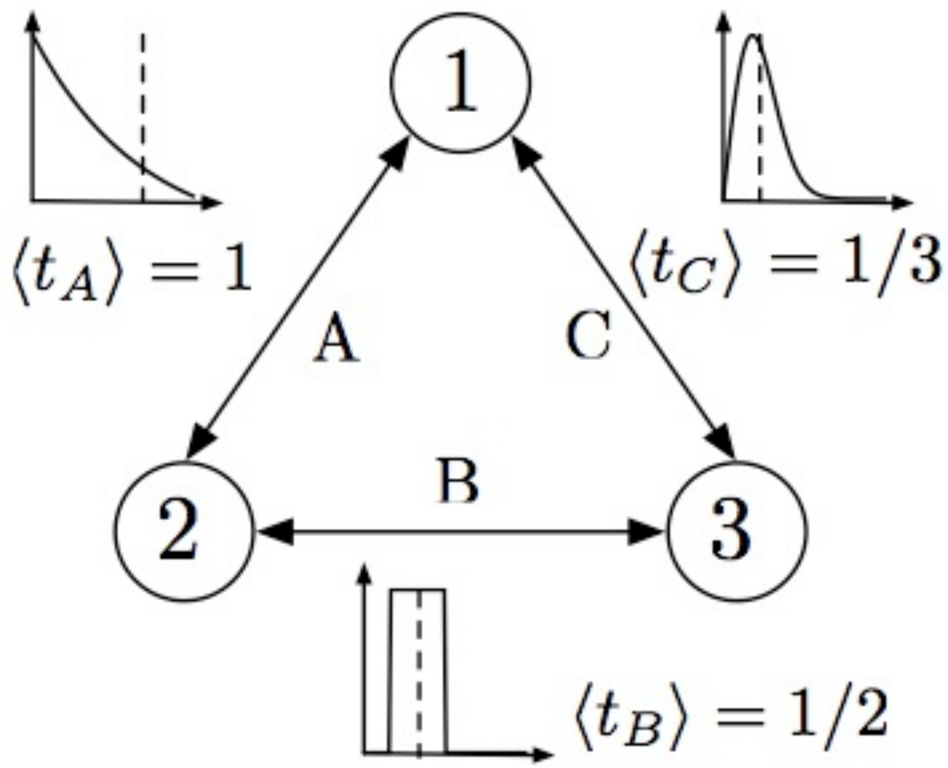
$$\frac{dn_i}{dt} = \sum_j \lambda_{ij} n_j(t) - \Lambda_i n_i(t) \ . \qquad (23)$$

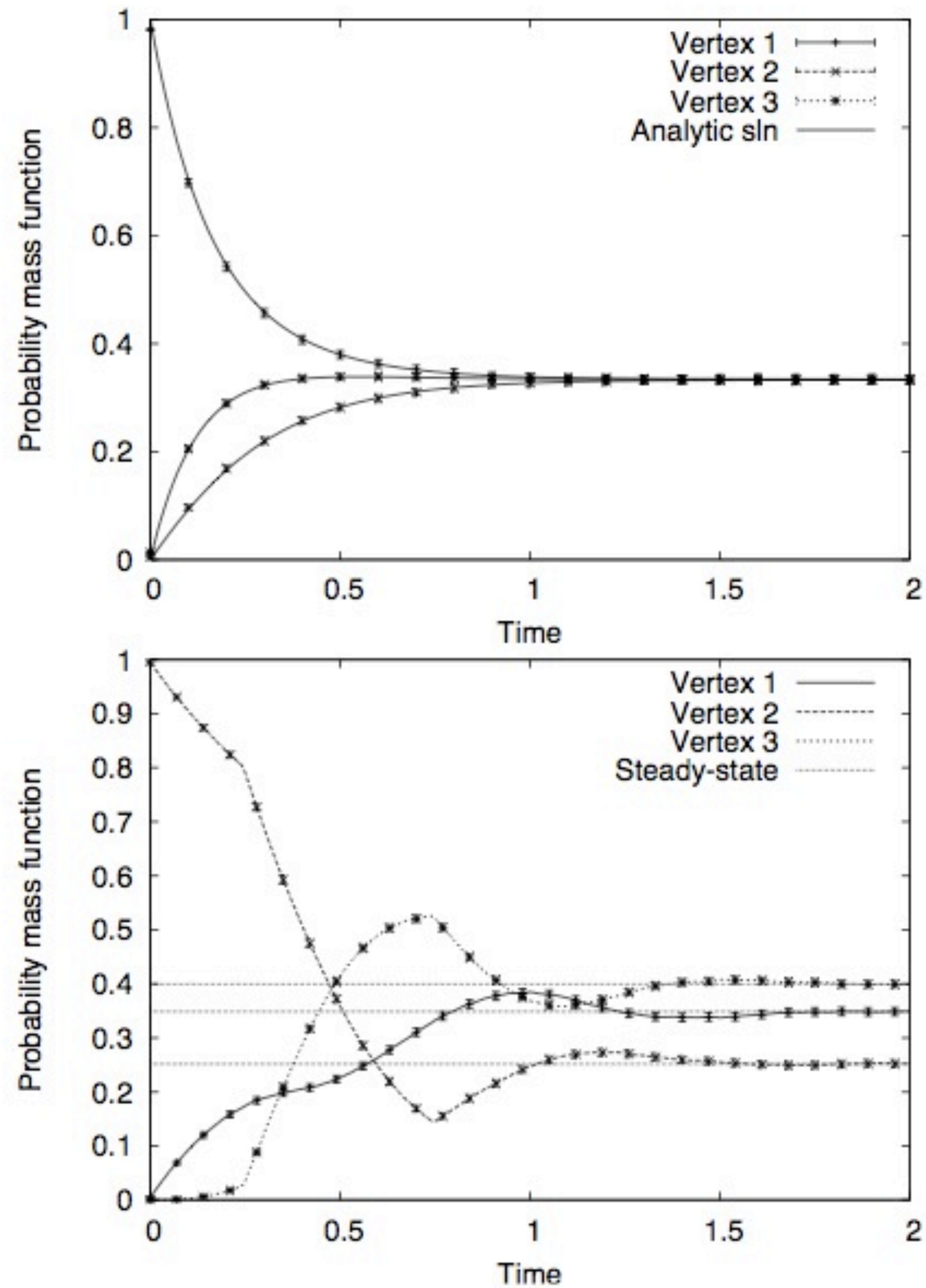
This dynamical process is driven by the combinatorial Laplacian $L_{ij} = \lambda_{ij} - \Lambda_i \delta_{ij}$ of the underlying weighted network.

Generalized Master Equations for Non-Poissonian Dynamics on Networks, T.H., M.A.P. and R.L.

Random walks on stochastic networks

Steady-state solution

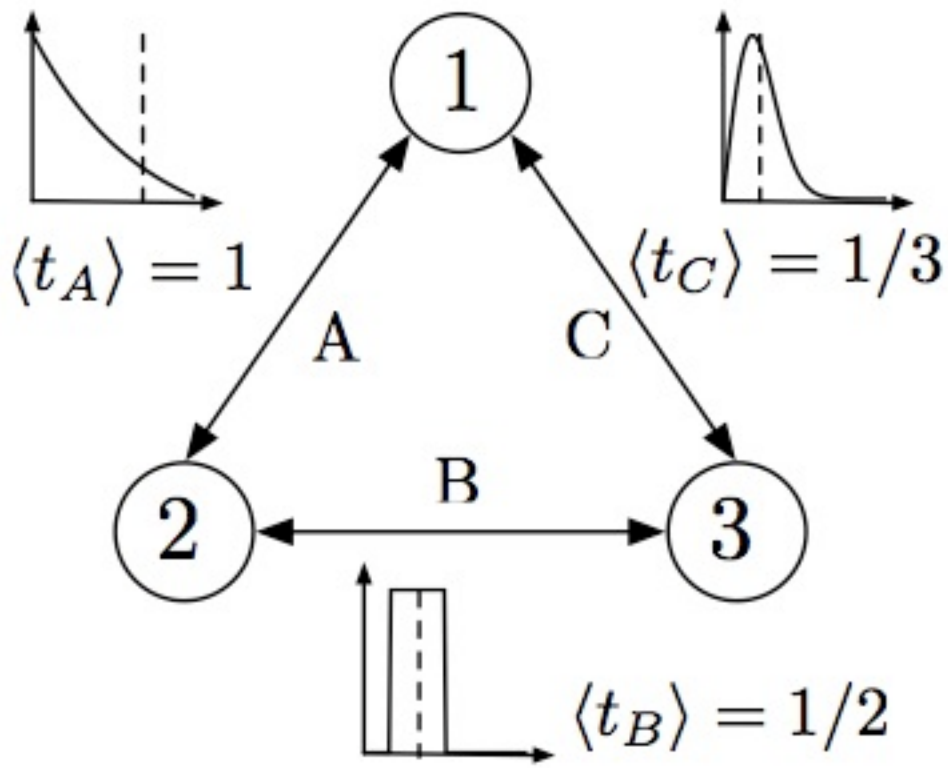




Generalized Master Equations for Non-Poissonian Dynamics on Networks, T.H., M.A.P. and R.L.

Random walks on stochastic networks

Steady-state solution only steady at long times (memory)



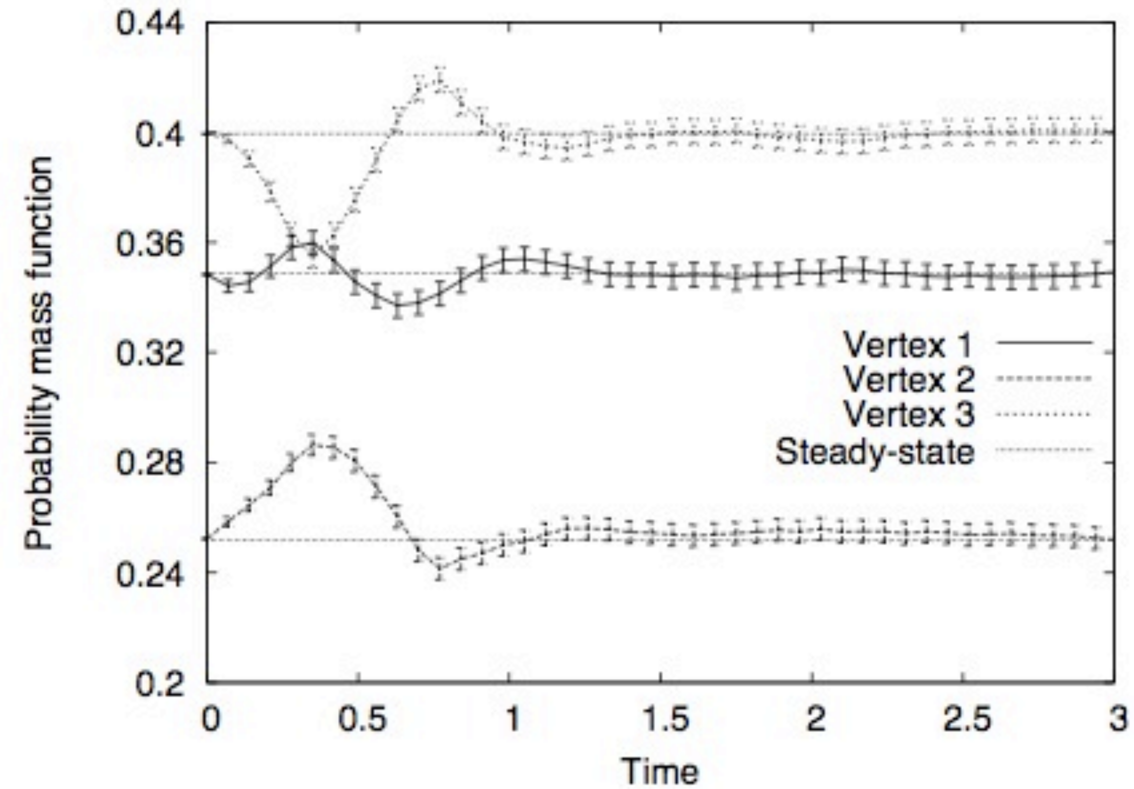


FIG. 4: Random-walker density on each node of the graph illustrated in Fig. 2 as a function of time when the initial condition is the steady-state solution. The system exhibits transient dynamics before returning to its steady-state solution due to the time-dependent nature of the dynamics. The error-bars represent the 5σ confidence interval.

Generalized Master Equations for Non-Poissonian Dynamics on Networks, T.H., M.A.P. and R.L.

Conclusion

Importance of teleportation on random walk metrics

Effect of the time evolution of the network on diffusion